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Anatomy of two holographic renormalization group flows

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Abstract: We derive and solve a subset of the fluctuation equations about two domain wall solutions of $D = 5$, $\mathcal{N} = 8$ gauged supergravity. One solution is dual to $D = 4$, $\mathcal{N} = 4$ SYM theory perturbed by an $\mathcal{N} = 1$, SO(3)-invariant mass term and the other to a Coulomb branch deformation. In the first case we study all SO(3)-singlet fields. These are assembled into bulk multiplets dual to the stress tensor multiplet and to the $\mathcal{N} = 1$ chiral multiplets $\mathrm{Tr}\,\Phi^2$ and $\mathrm{Tr}\,W^2$, the former playing the role of anomaly multiplet. Each of these three multiplets has a distinct spectrum of "glueball" states. This behavior is contrasted with the Coulomb branch flow in which all fluctuations studied have a continuous spectrum above a common mass gap, and spontaneous breaking of conformal symmetry is driven by a bulk vector multiplet. R-symmetry is preserved in the field theory, and correspondingly the bulk vector is dual to a linear anomaly multiplet. Generic features of the fluctuation equations and solutions are emphasized. For example, the transverse traceless modes of all fields in the graviton multiplet can be expressed in terms of an auxiliary massless scalar, and gauge fields associated with R-symmetry have a universal effective mass.

Keywords: AdS-CFT Correspondance, Supergravity Models.

Contents

1. Introduction

It is quite remarkable that the AdS/CFT correspondence [1] describes the large N strong coupling limit of 4D gauge theories with both exact and broken conformal symmetry. There is by now a large literature discussing the situation in which the renormalization group flow of field theory couplings is, in principle, described by supergravity; it is difficult to be complete, but a representative sample of this work may be roughly divided into the categories of i) early examples [2]–[8], ii) explicit domain wall solutions of $D = 5$ and $D = 10$ supergravities [9]–[32], and iii) holographic formulation of renormalization group equations [33]–[40].

A holographic RG flow is in general described by a solution of a $D \geq 5$ supergravity theory whose symmetry group is the 4D Poincaré group or a supersymmetric extension thereof. The solutions of linearized fluctuation equations in the bulk geometry can, in principle, be used to calculate correlation functions of operators in the dual boundary field theory. Our purpose is to derive and solve these fluctuation equations in examples of the two main cases of physical interest, the case where the bulk flow is dual to a massive deformation of $\mathcal{N} = 4$ super Yang-Mills (SYM) theory [16] and the case where it is dual to a Coulomb branch vacuum of $\mathcal{N} = 4$ SYM [10, 11].

Both bulk geometries were derived from the maximal gauged $\mathcal{N} = 8, D = 5$ supergravity [41, 42]. We select for study the fluctuations of fields in the bulk graviton multiplet together with other multiplets to which the graviton and its superpartners are coupled. This allows us to explore the way in which conformal and R-symmetries are broken in the boundary theory and to illuminate the relation between bulk and boundary supersymmetry. We also want to contrast the behavior in the two situations of operator and Coulomb branch deformations. The fluctuation equations initially extracted from $\mathcal{N} = 8$ supergravity are quite complicated, but we are able to manipulate them into simpler "universal" forms which appear to be more general than the specific flows from which they were obtained. All fluctuation equations are effectively hypergeometric, and their solutions give the essential features of the "glueball spectra" in both flows we study.

Most of our attention will focus on the flow proposed by Girardello, Petrini, Porrati and Zaffaroni (GPPZ) [16], where a kink solution was found involving a scalar field $m(r)$ which turns on a common mass for the three $\mathcal{N} = 1$ chiral multiplets of $\mathcal{N} = 4$ SYM theory. The massive theory flows toward pure $\mathcal{N} = 1$ SYM at large distance. The field theory interpretation of [16] poses several problems, and there may be inherent difficulties in a 5D approach in which curvature singularities are endemic. Indeed very interesting non-singular descriptions of the massive theory, based on 3- and 5-branes in $D = 10$ type-IIB supergravity have appeared [23, 28, 29]. Also recently, the metric and dilaton/axion fields of the "lift" of the GPPZ flow to ten dimensions were obtained [27], suggesting new physical features. In any case, the

GPPZ flow is a mathematically consistent example of holography, and the boundary behavior of bulk fields agrees with the field theory interpretation made in [16]. This should be enough to illuminate the behavior of the correlators of important operators such as the stress tensor and symmetry currents.

The breakdown of conformal symmetry in the massive field theory is reflected in supergravity by mixing of the graviton trace h^μ_μ with the scalar fluctuation $\tilde{m}$, as discussed in [43, 44]. In addition to $h_{\mu\nu}$, the gravity multiplet contains a symplectic Majorana gravitino $\psi^a_\mu, a = 1, 2$, and a $\mathrm{U}(1)_R$ gauge field B_μ. Because of broken superconformal symmetry and broken R-symmetry respectively, ψ^a_μ mixes with a spin-1/2 field ξ^a, and B_μ mixes with a scalar β. The fields $\xi^a, \tilde{m}, \beta$ and two more scalars span a bulk $\mathcal{N} = 2$ hypermultiplet dual to the anomaly multiplet in the field theory. There is a second, inert hypermultiplet containing the dilaton dual to the operator $\mathrm{Tr}\, F^2 + \cdots$ and its SUSY partners.

The graviton multiplet and these two hypermultiplets contain all bulk fields which are singlets of the SO(3) flavor symmetry preserved by the GPPZ flow, and we will obtain and solve the fluctuation equations for all these modes. Although these singlet fields decouple from the rest of the $\mathcal{N} = 8$ supergravity theory, their mutual interactions are still rather intricate. In particular, the dynamics of the 8 real scalars determines a nonlinear σ-model on the quaternionic manifold $G_{2(2)}/\,\mathrm{SO}(4)$. Even at the level at which we work, which is exact in 2 of the 8 scalars and bilinear in all other fields, the extraction of the field equations for the singlet sector from the full $\mathcal{N} = 8$ theory is a complex technical task that would be difficult to carry out without extensive use of an algebraic manipulation program. We have used Mathematica to compute the full scalar action for the SO(3) singlet sector. In particular, we have obtained the exact potential for all 8 scalar fields; the calculation was feasible using the so-called solvable parameterization of the scalar coset (see, e.g., [45]).

The fluctuations of the transverse traceless components of h_{ij}, the scalar m and another scalar σ have been obtained previously [43, 44, 38], and it is known that their fluctuation equations can be transformed to hypergeometric form with 3 distinct hypergeometric solutions, whose asymptotics in turn determine the discrete spectrum of dual field theory states. We show explicitly that all SO(3)-singlet boson and fermion fluctuations involve the same three hypergeometric functions, corresponding to three distinct glueball spectra for the dual operators, as we now summarize:

For the $\mathcal{N} = 1$ supercurrent multiplet $\mathcal{J}_{\alpha\dot{\alpha}} = \mathrm{Tr}(W_\alpha \bar{W}_{\dot{\alpha}} + \cdots)$ dual to the transverse components of the bulk supergravity multiplet $\{h_{\mu\nu}, \psi^{1,2}_\mu, B_\mu\}$, we have states with momenta:

$$(pL)^2 = 4(n+2)^2 \qquad n = 0, 1, 2, \ldots \tag{1.1}$$

For the $\mathcal{N} = 1$ chiral anomaly multiplet $\mathcal{A} = \mathrm{Tr} \sum_{i=1}^3 (\Phi_i^2)$ dual to the active hypermultiplet $\{\rho, \xi^{1,2}, \tilde{m}\}$:

$$(pL)^2 = 4(n+1)(n+2) \qquad n = 0, 1, 2, \ldots \tag{1.2}$$

For the $\mathcal{N}=1$ chiral "lagrangian" multiplet $\mathcal{S}=\mathrm{Tr}(W^\alpha W_\alpha+\cdots)$ dual to the dilaton hypermultiplet $\{\sigma,\xi^{3,4},\tau\}$,

$$(pL)^2 = 4n(n+3) \qquad n = 0,1,2,\ldots\,, \tag{1.3}$$

including a zero-mass pole for the lowest component operators dual to σ.

This pattern agrees with physical expectations, but it emerges in a subtle way from the interwoven symmetries and dynamics of the bulk supergravity theory. The dilaton and axion fields τ are treated correctly for the first time; to do this forces us to confront the complexity of the $G_{2(2)}/\,\mathrm{SO}(4)$ coset. In particular we find that although the transverse traceless modes in the supergravity multiplet can be conveniently written in terms of an auxiliary scalar field, there is no physical scalar field with the same modes in the theory.[1]

We then go on to explore the way spontaneously broken conformal symmetry is realized in a supergravity background by examining the fermion and vector sectors of a Coulomb branch flow, the "$n=2$" configuration corresponding to a disc of D3-branes found in [10, 11]. The active scalar, denoted by $\varphi(r)$, turns on an expectation value for a real component of the scalar bilinear $\mathrm{Tr}(X^2)$ in the $20'$ representation of SU(4). This background preserves $\mathcal{N}=4$ supersymmetry and $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ R-symmetry. All field theory operators examined possess a continuous spectrum above a common mass gap.

Examining the fluctuation equations, we find that the gravitino/spin-1/2 sectors have an identical structure to the GPPZ case, as was already known to occur with $h^\mu_\mu/\tilde{\varphi}$. The graviphoton remains massless, since the U(1) R-current is conserved; however, it couples to the active scalar background via a modified kinetic term. This difference has its origin in the fact that φ is real and in a vector multiplet, as contrasted with the complex m which sits in a hypermultiplet. This vector multiplet is naturally associated with a *linear* anomaly multiplet in the dual field theory, as is known to arise in non-conformal theories with preserved R-symmetry [47]. Thus two examples we consider demonstrate that the bulk multiplet containing the active scalar is intimately linked to the type of anomaly multiplet arising in the field theory. Other recent work on the identification between bulk and boundary supermultiplets can be found in [48]. Finally, we show that all although massless vectors in the Coulomb branch flow possess modified kinetic terms, their equations of motion can be transformed to eliminate these in favor of the common mass term $m_B^2=-2A''$ with the same form as for the GPPZ graviphoton.

In section 2 we establish notation and review the metric, connections, and Killing spinors for domain walls in 5 dimensions. The GPPZ and Coulomb branch solutions are presented. In section 3 we discuss the extraction of actions and transformation

[1] As discussed at the end of section 3.1, we believe that the dilaton was not treated correctly in [20]. Incorrect statements about a massless scalar fluctuation in the GPPZ flow appear in [43, 46].

rules for the SO(3)-singlet fields from the $\mathcal{N} = 8$ theory and present results. In section 4 we show that the transverse traceless modes of all fields in the graviton multiplet can be written in terms of an auxiliary massless scalar field. In section 5 we summarize and synthesize previous results of [43, 44] for the coupled $h^\mu_\mu/\tilde{m}$ sector and present the 2-point function for the Coulomb branch case. Section 6 is devoted to the decoupling and solution of fluctuation equations for vector and scalar fields. The analogous discussion for the fermion sector is given in section 7, in which we also verify the Bianchi identity for the gravitino equation of motion. The fermion and vector fluctuations of the Coulomb branch flow are discussed in section 8, and similarities and differences to the GPPZ case noted. Although the results of section 3 are the basis of the fluctuation equations solved in later sections, these sections are largely self-contained and can be understood without a detailed reading of section 3.

The next step in this investigation will involve the systematic application of results for bulk field fluctuations to obtain 2-point correlation functions of the dual operators in the field theory and to study physical implications. This question was addressed in [43, 44], but there is still need to clarify the extraction of field theory information from the supergravity fluctuations. For example, a procedure to obtain correlation functions of the stress tensor which is invariant under gauge choices made in treating bulk fluctuations is desirable. We hope to report on this soon.

2. Review of GPPZ and Coulomb branch flows

In this paper we shall be concerned with two backgrounds of 5D maximally supersymmetric gauged supergravity, exemplifying the two distinct types of dual RG flows: the operator deformation, which modifies the lagrangian of the dual field theory, and the Coulomb deformation, which only modifies the vacuum of the theory. In the examples we study there is one "active" scalar field $\phi(r)$ which depends on the radial coordinate of the geometry.

In our conventions, the 5D Newton constant is such that $\kappa_5^2 = 2$ and the coupled gravity/scalar action is

$$S = \int d^5x \sqrt{g} \left[-\frac{1}{4}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \right], \tag{2.1}$$

where we use $(+----)$ signature. The background geometry always has domain wall form

$$ds^2 = e^{2A(r)}\,(\eta_{ij}\,dx^i dx^j) - dr^2\,. \tag{2.2}$$

The frames, nonvanishing Christoffel and spin connections, and curvature tensors derived from the metric (2.2) are

$$\begin{aligned}
e^{\hat{k}} &= e^{A} dx^{k} , & e^{\hat{r}} &= dr , & & \\
\Gamma^{r}_{ij} &= A' g_{ij} , & \Gamma^{i}_{jr} &= \Gamma^{i}_{rj} = A' \delta^{i}_{j} , & \omega_{j}{}^{\hat{k}\hat{r}} &= -\omega_{j}{}^{\hat{r}\hat{k}} = -A' e^{A} \delta^{\hat{k}}_{j} , \\
R_{ijkl} &= (g_{ik} g_{jl} - g_{il} g_{jk}) A'^{2} , & R_{irjr} &= -g_{ij} (A'^{2} + A'') , & & \\
R_{ij} &= (4A'^{2} + A'') g_{ij} , & R_{rr} &= -4A'^{2} - 4A'' , & R &= 20A'^{2} + 8A'' .
\end{aligned} \tag{2.3}$$

Here and throughout we use i, j as 4D curved indices, μ, ν as 5D curved indices and $\hat{k}, \hat{l}$ as 4D Lorentz indices.

In the boundary limit $r \to \infty$ the geometry approaches that of AdS_5, with asymptotic scale factor $A(r) \to r/L$, while the active scalar, assumed dual to a field theory operator $\mathcal{O}_\phi$ of scale dimension Δ, has the asymptotic form $\phi(r) \sim e^{(\Delta-4)r}$ for operator perturbations and $\phi(r) \sim e^{-\Delta r}$ for vacuum deformations.[2] Far from the boundary both background geometries we consider present curvature singularities where $e^{2A(r)} \to 0$. Although this formally indicates a breakdown of the supergravity description, we shall proceed by choosing the solutions of all fluctuation equations to be regular at the singularity, which is the established procedure. The singularities of both flows we consider satisfy the acceptability criterion formulated by Gubser [49].

The spacetime symmetry of the general domain wall is the 4D Poincaré group. The special case when $A(r) = r/L$ and $\phi(r) = const$ corresponds to an exact AdS_5 geometry with an additional 5 isometries, dual to scale and special conformal transformations in the field theory.

The backgrounds we study are supersymmetric, which means that they possess Killing spinors, zero modes of the spinor transformation rules of the bulk supergravity theory [50]. To discuss these we recall that the gauged $\mathcal{N} = 8$ theory contains **8** symplectic Majorana gravitino fields $\psi_{\mu a}$, with the USp(8) index $a = 1, \ldots, 8$ raised and lowered by the symplectic metric Ω_{ab}, and **48** spinor fields χ_{abc} whose transformation rules are [51]

$$\delta\psi_{\mu a} = D_\mu \epsilon_a - \frac{g}{6} W_{ab}(\phi) \gamma_\mu \epsilon^b , \tag{2.4}$$

$$\delta\chi_{abc} = \left[\sqrt{2} \gamma^\mu P_{\mu abcd}(\phi) - \frac{g}{\sqrt{2}} A_{dabc}(\phi) \right] \epsilon^d , \tag{2.5}$$

where $g = 2/L$ is the bulk gauge coupling constant, and the USp(8) tensors W_{ab}, A_{abcd}, and $P_{\mu abcd}$ are complicated functions of the 42 scalar fields of the theory. Their expressions simplify vastly after group theoretic analysis is used to truncate the equations of motion of the theory to subsets of one or two scalars, a process first done in the context of 5D RG flows in [9]. On such a scalar subset one examines the symplectic eigenvalue problem for the symmetric matrix W_{ab}. Let $W(\phi)$ denote the

[2] For the case $\Delta = 2$, the scalings are re^{-2r} and e^{-2r}, respectively.

eigenvalue on a given two dimensional eigenspace spanned by a symplectic Majorana pair ϵ_1, ϵ_2. On this subspace, the gravitino transformation rule reduces to

$$\delta\psi_{\mu a} = D_\mu \epsilon_a - \frac{g}{6} W(\phi) \gamma_\mu \Omega_{ab}\, \epsilon^b \,. \tag{2.6}$$

The first condition on Killing spinors is that they are zero modes of (2.6). This is achieved by imposing the first order flow equation [9],

$$A'(r) = -\frac{g}{3} W(\phi(r))\,, \tag{2.7}$$

which relates the scale factor and scalar field profile in the background. Killing spinors then take the form

$$\epsilon = e^{A(r)/2} \eta^{(0)}\,, \qquad i\gamma^r \eta^{(0)} = \eta^{(0)}\,, \tag{2.8}$$

where $\eta^{(0)}$ is a constant complex superposition $\eta^{(0)} = \eta_1^{(0)} + i\eta_2^{(0)}$ on the original eigenspace; the chirality condition indicates the Killing spinor contains a 4D constant Weyl spinor, so it is 4D Poincaré supersymmetry which is naturally associated with the domain wall. In the complex notation the Killing spinor (2.8) satisfies

$$D_\mu \epsilon = i\frac{g}{6} W \gamma_\mu \epsilon \,. \tag{2.9}$$

The second condition on Killing spinors is that they are also zero modes of (2.5). This is assured if, on the same eigenspace, the tensors $P_{\mu abcd}$ (linear in derivatives of the active scalar $\phi(r)$) and A_{abcd} simplify, so that (2.5) reduces to the flow equation [9],

$$\phi'(r) = \frac{g}{2} \frac{\partial W(\phi)}{\partial \phi} \,. \tag{2.10}$$

The flow equations (2.7) and (2.10) are easily generalized to solutions with several active scalars and to the case of a non-trivial σ-model metric. In particular (2.10) is just a gradient flow equation for the function $W(\phi)$, which is called the superpotential for the active scalars because it is related to the potential $V(\phi)$ on the reduced scalar subspace of the flow by

$$V(\phi) = g^2 \left[\frac{1}{8} \left(\frac{\partial W(\phi)}{\partial \phi} \right)^2 - \frac{1}{3} W(\phi)^2 \right] . \tag{2.11}$$

The significant feature of the first order flow equations is that any solution $\{\phi(r), A(r)\}$ is guaranteed by supersymmetry to be a solution of the second order field equations of the action (2.1). In the case of one active scalar it is usually straightforward to solve (2.7, 2.10) explicitly.

We have seen that a domain wall solution of (2.7), (2.10) generically has $\mathcal{N} = 1$ 4D Poincaré symmetry. Additional supersymmetries appear if the eigenvalue $W(\phi)$

of W_{ab} is degenerate, and it is known that the Coulomb branch flow we study has maximal $\mathcal{N} = 4$ Poincaré supersymmetry. Conformal supersymmetry occurs in the boundary field theory when the bulk geometry is exactly AdS_5. In this case there are additional Killing spinors of the form [52]

$$\epsilon_{SC} = (1 - iA' x^j \gamma_j)\, e^{-A/2}\, \zeta^{(0)}\,, \qquad i\gamma^r \zeta^{(0)} = -\zeta^{(0)}\,, \tag{2.12}$$

with $\zeta^{(0)}$ a constant 4D Weyl spinor.

2.1 The GPPZ flow

We shall primarily focus on the geometry first considered by GPPZ [16] as a candidate dual description of a confining gauge theory. This background represents an operator deformation of $\mathcal{N} = 4$ SYM, where the active scalar, m, corresponds to the addition of equal masses for the three chiral superfields in the $\mathcal{N} = 1$ language.

The flow breaks the gauge symmetry from SU(4) to SO(3) and the supersymmetry from 32 supercharges to 4. As will be discussed at length in the next section, the set of SO(3)-singlet fields encompasses the gravity multiplet and two hypermultiplets, the “active” hyper which contains m, as well as the dilaton hypermultiplet, also containing the scalar σ.[3]

The superpotential involving both scalars is

$$W(m, \sigma) = -\frac{3}{4}\left[\cosh\left(\frac{2m}{\sqrt{3}}\right) + \cosh(2\sigma)\right], \tag{2.13}$$

and the $\sigma = 0$ background that we study is described by

$$m(r) = \frac{\sqrt{3}}{2}\log\frac{1 + e^{-r/L}}{1 - e^{-r/L}}\,, \qquad A(r) = \frac{1}{2}\left[\frac{r}{L} + \log\left(2\sinh\frac{r}{L}\right)\right], \tag{2.14}$$

containing a singularity at finite proper distance located at $r = 0$. The variable that proves convenient for solving the various fluctuation equations is [43]

$$u \equiv 1 - e^{-2r/L}\,, \tag{2.15}$$

in terms of which

$$W = -\frac{3}{2u}\,, \qquad \frac{\partial W}{\partial m} = -\sqrt{3}\,\frac{\sqrt{1-u}}{u}\,, \qquad e^{2A} = \frac{u}{1-u}\,, \quad \frac{du}{dr} = \frac{2}{L}(1-u)\,. \tag{2.16}$$

In the u-variable, the boundary is at $u = 1$ and the singularity at $u = 0$.

[3]The authors of [16] additionally considered a family of flows with both m and σ active. We restrict to the $\sigma = 0$ case here, although we mention the $\sigma \neq 0$, $m = 0$ background in section 8.4.

2.2 The Coulomb branch flow

The other background we will consider is a Coulomb branch deformation, studied in [10, 11] and called the $n = 2$ flow in [10]. It has a 10D lift to a geometry surrounding a 2D disk of D3-branes, and preserves the symmetry $\mathrm{SO}(4) \times \mathrm{SO}(2) \cong \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. The sixteen supercharges dual to ordinary supersymmetries (2.8) are all preserved, while the superconformal supercharges (2.12) are broken.

The superpotential is

$$W(\varphi) = -e^{-2\varphi/\sqrt{6}} - \frac{1}{2}\, e^{4\varphi/\sqrt{6}}\,, \tag{2.17}$$

and $\varphi \to -\infty$ as one approaches the interior. The convenient variable for studying fluctuations in this flow is related to the field φ by [10]

$$v \equiv e^{\sqrt{6}\,\varphi}\,. \tag{2.18}$$

The boundary is at $v = 1$, and a curvature singularity appears at $v = 0$. The solution for the flow is given by

$$W = -\frac{1}{2}\,\frac{v+2}{v^{1/3}}\,, \quad \frac{\partial W}{\partial \varphi} = \frac{2}{\sqrt{6}}\,\frac{1-v}{v^{1/3}}\,, \quad e^{2A} = \frac{\ell^2}{L^2}\,\frac{v^{2/3}}{1-v}\,, \quad \frac{dv}{dr} = \frac{2}{L} v^{2/3}(1-v)\,. \tag{2.19}$$

The length scale ℓ is the radius of the disc of D3-branes.

One difference between the geometries of the two flows we study is the behavior of radial null geodesics ($dt = -e^{-A(r)}dr$) departing any interior point. These reach the singularity in finite time t for the GPPZ flow but take infinite time for the Coulomb branch flow.

3. The $\mathrm{SO}(3)$-invariant sector of $\mathcal{N} = 8$ gauged supergravity in five dimensions

It was shown in [27] that the SO(3)-invariant sector of the $\mathcal{N} = 8$ supergravity in five dimensions is described by an $\mathcal{N} = 2$ gauged supergravity coupled to two hypermultiplets with the scalar fields parameterizing the quaternionic manifold

$$\mathcal{Q}_0 = \frac{G_{2(2)}}{\mathrm{SO}(4)}\,. \tag{3.1}$$

In this section we present the linearized action and supersymmetry transformation rules of this theory, which are derived by performing an explicit truncation of the $\mathcal{N} = 8$ supergravity to the SO(3)-singlet fields.[4] Our starting point is the $\mathcal{N} = 8$

[4]The quaternionic coset $G_{2(2)}/\,\mathrm{SO}(4)$ was first studied from a different vantage point in [53] and later in [54]. In particular, the latter reference contains an explicit realization of the $G_{2(2)}$ generators. The gauged theory which we obtain here at the linearized level should be a particular case of gauged $\mathcal{N} = 2$ $D = 5$ supergravity coupled to matter recently constructed in [55]. However, we have not derived a precise mapping between the two cases.

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theory [41, 42] as formulated in [51], which the reader should consult for conventions and further details (see, also, [9, appendix A]).

We begin our discussion with a fairly detailed treatment of the scalar sector, which requires an explicit parameterization of the scalar manifold, $\mathcal{Q}_0$, and for which the linearized action is by far the most difficult to extract. This is followed by a summary of results for the other sectors, and finally the linearized supersymmetry transformations.

3.1 The scalar fields

Recall that the scalar fields of the $\mathcal{N}=8$ gauged supergravity in five dimensions are given by a nonlinear σ-model on the noncompact coset manifold $E_{6(6)}/\,\mathrm{USp}(8)$. The basic object here is the coset representative, $(\mathcal{V}^{IJ\,ab}, \mathcal{V}_{I\alpha}{}^{ab})$, called the "27-bein," which is a matrix of $E_{6(6)}$ in a 27-dimensional real representation. The **27** is conveniently written in the $\mathrm{SL}(6,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ basis as $(z_{IJ}, z^{I\alpha})$, where $I,J=1,\ldots,6$ and $\alpha,\beta=1,2$, and $z_{IJ}=-z_{JI}$ [51]. The corresponding infinitesimal action of $E_{6(6)}$ is then given by (see, [56] and, in particular, [51, appendix A])

$$\begin{aligned} \delta z_{IJ} &= -\Lambda^K{}_I z_{KJ} - \Lambda^K{}_J z_{IK} + \Sigma_{IJK\beta} z^{K\beta}\,, \\ \delta z^{I\alpha} &= \Lambda^I{}_K z^{K\alpha} + \Lambda^\alpha{}_\beta z^{I\beta} + \Sigma^{KLI\alpha} z_{KL}\,, \end{aligned} \tag{3.2}$$

where $\Lambda^I{}_J$, $\Lambda^\alpha{}_\beta$ correspond to $\mathrm{SL}(6,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{R})$ transformations respectively, and $\Sigma_{IJK\alpha}=\frac{1}{6}\epsilon_{IJKLMN}\epsilon_{\alpha\beta}\Sigma^{LMN\beta}$. The gauge group SO(6) is generated by $\Lambda^I{}_J = -\Lambda^J{}_I$.

Various SU(2) invariant truncations of the theory can be obtained by selecting a particular SU(2) subgroup of the SO(6) gauge group and then restricting to the singlet fields [57, 9, 27]. The truncation relevant for the discussion of the GPPZ flow is obtained by taking the obvious maximal SO(3) subgroup that is diagonal in $\mathrm{SO}(3)\times\mathrm{SO}(3)\subset\mathrm{SO}(6)$, where the first SO(3) acts on the indices $I=1,2,3$ and the second one acts on $I=4,5,6$. The branching rules that describe this embedding $\mathrm{SO}(3)\subset\mathrm{SO}(6)\cong\mathrm{SU}(4)$ are $\mathbf{4}\to\mathbf{3}\oplus\mathbf{1}$ and $\mathbf{6}\to\mathbf{3}\oplus\mathbf{3}$.

As discussed in detail in [57, 9, 27], the scalar manifold of a truncated theory is given by a coset C/K, where C is the maximal subgroup of $E_{6(6)}$ that commutes with the invariance subgroup and K is its maximal compact subgroup. In the present case the Lie algebra of C can be constructed explicitly as follows:

There are two obvious contributions coming from the $\mathrm{SL}(2,\mathbb{R})\subset\mathrm{SL}(6,\mathbb{R})$ transformations parametrized by

$$(\Lambda^I{}_J) = \begin{pmatrix} s_3\,\mathbf{1} & (s_1-s_2)\,\mathbf{1} \\ (s_1+s_2)\,\mathbf{1} & -s_3\,\mathbf{1} \end{pmatrix}, \tag{3.3}$$

where $\mathbf{1}$ is a 3×3 unit matrix, and from the $\mathrm{SL}(2,\mathbb{R})_\tau$ transformations

$$(\Lambda^\alpha{}_\beta) = \begin{pmatrix} a_3 & a_1-a_2 \\ a_1+a_2 & -a_3 \end{pmatrix}, \tag{3.4}$$

corresponding to "the dilaton/axion" field τ in five dimensions. The remaining contribution arises from the transformations parametrized by the Σ-tensor that are invariant under SO(3). To this end define

$$\begin{aligned} X_{(1)}^{IJK} &= \delta_{123}^{IJK}\,, \\ X_{(2)}^{IJK} &= \delta_{423}^{IJK} + \delta_{153}^{IJK} + \delta_{126}^{IJK}\,, \\ X_{(3)}^{IJK} &= \delta_{156}^{IJK} + \delta_{426}^{IJK} + \delta_{453}^{IJK}\,, \\ X_{(4)}^{IJK} &= \delta_{456}^{IJK}\,, \end{aligned} \tag{3.5}$$

where $\delta_{MNP}^{IJK} = \frac{1}{3!}(\delta_M^I \delta_N^j \delta_P^K + \text{permutations})$ and set

$$\Sigma^{IJK\alpha} = 3! \sum_{i=1}^{4} \tau^{i\alpha}\, X_{(i)}^{IJK}\,. \tag{3.6}$$

One can check that the 14-parameter transformations (3.3), (3.4) and (3.6) indeed generate the $G_{2(2)}$ subalgebra of $E_{6(6)}$ [27]. We will demonstrate this explicitly by constructing a standard Cartan basis (see, e.g., [58]).

Let us first choose the two simple roots, α_1 and α_2, of $G_{2(2)}$ as

$$\alpha_1 = \left(-\sqrt{\frac{2}{3}}, 0\right), \qquad \alpha_2 = \left(\sqrt{\frac{3}{2}}, \sqrt{\frac{1}{2}}\right), \tag{3.7}$$

so that

$$(\alpha_1, \alpha_1) = \frac{2}{3}\,, \qquad (\alpha_2, \alpha_2) = 2\,, \qquad (\alpha_1, \alpha_2) = -1\,. \tag{3.8}$$

The set of positive roots, Δ_+, consists of

$$\alpha_1\,, \quad \alpha_1 + \alpha_2\,, \quad 2\alpha_1 + \alpha_2\,, \quad \alpha_2\,, \quad 3\alpha_1 + \alpha_2\,, \quad 3\alpha_1 + 2\alpha_2\,, \tag{3.9}$$

where the first three roots are short and the other three are long. The set of all roots is then $\Delta = \Delta_+ \cup (-\Delta_+)$.

The Cartan basis of $G_{2(2)}$ consists of two Cartan generators, H_1 and H_2, and the root generators, X_α, $\alpha \in \Delta$. (For a root $\alpha = k_1\alpha_1 + k_2\alpha_2$, we will also use the notation $X_{(k_1 k_2)} \equiv X_\alpha$.) For a maximally noncompact algebra, such as $G_{2(2)}$, the Cartan generators can be chosen to be noncompact [58]. It will be convenient here to take them along the m and σ coordinate of the extended GPPZ flow on the scalar manifold $\mathcal{Q}_0$. The generators H_1 and H_2 are given in terms of those of (3.3)–(3.6) in table 1. By diagonalizing the real matrices $\mathrm{ad}(H_i)$, $i = 1, 2$, we find the root generators X_α satisfying $[H_i, X_\alpha] = \alpha^i X_\alpha$. In table 1 we have also given explicitly the positive root generators, which we will need in the following. The remaining (negative) root generators are obtained similarly with the result that some coefficients change sign, see table 2. As a consistency check one may verify

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	H_1	H_2	$X_{(10)}$	$X_{(11)}$	$X_{(21)}$	$X_{(01)}$	$X_{(31)}$	$X_{(32)}$
s_1	0	0	0	0	$\frac{1}{4\sqrt{3}}$	$\frac{1}{4}$	0	0
s_2	0	0	$\frac{1}{4\sqrt{3}}$	0	0	0	0	$\frac{1}{4}$
s_3	0	0	0	$-\frac{1}{4\sqrt{3}}$	0	0	$-\frac{1}{4}$	0
a_1	0	0	0	0	$\frac{\sqrt{3}}{4}$	$-\frac{1}{4}$	0	0
a_2	0	0	$-\frac{\sqrt{3}}{4}$	0	0	0	0	$\frac{1}{4}$
a_3	0	0	0	$-\frac{\sqrt{3}}{4}$	0	0	$\frac{1}{4}$	0
τ^{11}	0	0	$\frac{1}{8\sqrt{6}}$	0	$\frac{1}{8\sqrt{6}}$	$\frac{1}{24\sqrt{2}}$	0	$-\frac{1}{24\sqrt{2}}$
τ^{12}	$\frac{1}{8\sqrt{3}}$	$-\frac{1}{24}$	0	0	0	0	$-\frac{1}{12\sqrt{2}}$	0
τ^{21}	$-\frac{1}{24\sqrt{3}}$	$-\frac{1}{24}$	0	$-\frac{1}{12\sqrt{6}}$	0	0	0	0
τ^{22}	0	0	$\frac{1}{24\sqrt{6}}$	0	$\frac{1}{24\sqrt{6}}$	$-\frac{1}{24\sqrt{2}}$	0	$\frac{1}{24\sqrt{2}}$
τ^{31}	0	0	$\frac{1}{24\sqrt{6}}$	0	$-\frac{1}{24\sqrt{6}}$	$\frac{1}{24\sqrt{2}}$	0	$\frac{1}{24\sqrt{2}}$
τ^{32}	$\frac{1}{24\sqrt{3}}$	$\frac{1}{24}$	0	$-\frac{1}{12\sqrt{6}}$	0	0	0	0
τ^{41}	$-\frac{1}{8\sqrt{3}}$	$\frac{1}{24}$	0	0	0	0	$-\frac{1}{12\sqrt{2}}$	0
τ^{42}	0	0	$\frac{1}{8\sqrt{6}}$	0	$-\frac{1}{8\sqrt{6}}$	$-\frac{1}{24\sqrt{2}}$	0	$-\frac{1}{24\sqrt{2}}$

Table 1: The Cartan and positive root generators of $G_{2(2)} \subset E_{6(6)}$.

that the generators in tables 1 and 2 can be block diagonalized in agreement with the branching rule for $G_{2(2)} \subset E_{6(6)}$, namely $\mathbf{27} \to 3(\mathbf{7}) \oplus 6(\mathbf{1})$.

The generators we have constructed are canonically normalized such that $\mathrm{Tr}(H_i H_j) = 2\delta_{ij}$, $\mathrm{Tr}(X_\alpha X_{-\alpha}) = 2$ with all other traces being zero, with the traces evaluated in the 7-dimensional fundamental representation of $G_{2(2)}$. The antihermitean combinations $(X_\alpha - X_{-\alpha})$, $\alpha \in \Delta_+$, span the compact subalgebra, while H_1 and H_2 together with the combinations $(X_\alpha + X_{-\alpha})$, $\alpha \in \Delta_+$, span the noncompact orthogonal complement.

The usefulness of the above construction for our purposes lies in the fundamental theorem on the Iwasawa decomposition of the maximally noncompact Lie groups (see, e.g., [58]), which implies that the scalar coset (3.1) can be analytically parametrized by real coordinates s_i, $i = 1, 2$, and t_α, $\alpha \in \Delta_+$, where an explicit mapping is given by the group elements

$$g(s_i, t_\alpha) = \exp\left(\sum_{i=1}^{2} s_i H_i\right) \exp\left(\sum_{\alpha\in\Delta_+} t_\alpha X_\alpha\right), \qquad s_i\,, t_\alpha \in \mathbb{R}\,. \tag{3.10}$$

Given an explicit representation of the generators, the group element (3.10) is easy to compute — the factor on the left is a product of commuting $O(1,1)$ "rotations", while the factor on the right is an element of a solvable group[5] and thus a matrix of finite degree polynomials in the t_α's.

[5] This is why the parameterization has been called a "solvable parameterization" in [45].

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	$X_{(-10)}$	$X_{(-1-1)}$	$X_{(-2-1)}$	$X_{(0-1)}$	$X_{(-3-1)}$	$X_{(-3-2)}$
s_1	0	0	$\frac{1}{4\sqrt{3}}$	$\frac{1}{4}$	0	0
s_2	$-\frac{1}{4\sqrt{3}}$	0	0	0	0	$-\frac{1}{4}$
s_3	0	$-\frac{1}{4\sqrt{3}}$	0	0	$-\frac{1}{4}$	0
a_1	0	0	$\frac{\sqrt{3}}{4}$	$-\frac{1}{4}$	0	0
a_2	$\frac{\sqrt{3}}{4}$	0	0	0	0	$-\frac{1}{4}$
a_3	0	$-\frac{\sqrt{3}}{4}$	0	0	$\frac{1}{4}$	0
τ^{11}	$\frac{1}{8\sqrt{6}}$	0	$-\frac{1}{8\sqrt{6}}$	$-\frac{1}{24\sqrt{2}}$	0	$-\frac{1}{24\sqrt{2}}$
τ^{12}	0	0	0	0	$\frac{1}{12\sqrt{2}}$	0
τ^{21}	0	$\frac{1}{12\sqrt{6}}$	0	0	0	0
τ^{22}	$\frac{1}{24\sqrt{6}}$	0	$-\frac{1}{24\sqrt{6}}$	$\frac{1}{24\sqrt{2}}$	0	$\frac{1}{24\sqrt{2}}$
τ^{31}	$\frac{1}{24\sqrt{6}}$	0	$\frac{1}{24\sqrt{6}}$	$-\frac{1}{24\sqrt{2}}$	0	$\frac{1}{24\sqrt{2}}$
τ^{32}	0	$\frac{1}{12\sqrt{6}}$	0	0	0	0
τ^{41}	0	0	0	0	$\frac{1}{12\sqrt{2}}$	0
τ^{42}	$\frac{1}{8\sqrt{6}}$	0	$\frac{1}{8\sqrt{6}}$	$\frac{1}{24\sqrt{2}}$	0	$-\frac{1}{24\sqrt{2}}$

Table 2: The negative root generators of $G_{2(2)} \subset E_{6(6)}$.

In the following we will use instead of s_1 and s_2 the corresponding fields of the extended GPPZ-flow [16], which are simply given by

$$m = \frac{1}{\sqrt{2}}\, s_1\,, \qquad \sigma = \frac{1}{\sqrt{2}}\, s_2\,. \tag{3.11}$$

We will also retain in all the formulae of this section the complete dependence on the two fields m and σ.

Once the proper parameterization of the scalar coset $\mathcal{Q}_0$ has been introduced, the computation of the kinetic action and the potential is a matter of a straightforward algebra that can be carried out on a computer. First one evaluates the group elements (3.10) in a convenient 7-dimensional representation and derives the kinetic action using the standard σ-model techniques. Then by a basis change one obtains the 27-bein fields $(\mathcal{V}^{IJ\,ab}, \mathcal{V}_{I\alpha}{}^{ab})$, which can be substituted directly into the formulae in [51]. Since several calculations of this type have already been discussed in the literature (see, e.g. [9]) we will skip the details here.

In the absence of vector fields the kinetic action of the scalar fields is simply given by

$$\mathcal{L}^0_K(m,\sigma;t_\alpha) = \frac{1}{2}\left[(\partial m)^2 + (\partial\sigma)^2\right] + \frac{1}{2}\sum_{\alpha\in\Delta_+} e^{2\alpha(m,\sigma)}(\partial t_\alpha)^2\,, \tag{3.12}$$

where $\alpha(m,\sigma) = \sqrt{2}(\alpha^1 m + \alpha^2\sigma)$ for a root α.

Using the solvable parameterization we have obtained a complete expression for the potential. The t_α-independent part of the potential reproduces the result in [16, 20, 27]:

$$
\begin{aligned}
\mathcal{P}^{(0)}(m,\sigma) = -\frac{3}{16}g^2 \bigg(2 - \frac{1}{4}\cosh(4\sigma) + \frac{1}{4}\cosh\left(\frac{4m}{\sqrt{3}}\right) + \\
+ \cosh\left(\frac{2m}{\sqrt{3}} + 2\sigma\right) + \cosh\left(\frac{2m}{\sqrt{3}} - 2\sigma\right)\bigg) .
\end{aligned}
\tag{3.13}
$$

The complete expression for the potential is too long to be reproduced here.[6] It is a sixth order polynomial in the t_α's with no first order terms. Since all we need here is the quadratic expansion in the transverse fields t_α about an arbitrary m and σ configuration, we now restrict to this order. Define

$$
E(k_1,k_2) = e^{\alpha(m,\sigma)} \qquad \text{where} \qquad \alpha = k_1\alpha_1 + k_2\alpha_2 \,. \tag{3.14}
$$

Then the quadratic term in the expansion of the potential is

$$
\begin{aligned}
\mathcal{P}^{(2)}(m,\sigma;t_\alpha) = & -\frac{g^2}{16}\Big(1 + 2E(-2,-2) + E(2,0) + 2E(4,2)\Big)t_{(10)}^2 - \\
& -\frac{g^2}{32}\Big(8 - 3E(-4,-2) + 6E(-2,0) - E(0,2) - \\
& \qquad - 8E(2,2) + 2E(6,4)\Big)t_{(11)}^2 - \\
& -\frac{g^2}{32}\Big(8 - 3E(-2,-2) + 6E(2,0) - 8E(4,2) - \\
& \qquad - E(6,2) + 2E(6,4)\Big)t_{(21)}^2 + \\
& +\frac{3g^2}{32}\Big(E(-6,-2) - 4E(-2,0) - E(2,2)\Big)t_{(01)}^2 + \\
& +\frac{3g^2}{32}\Big(E(0,-2) - 4E(2,0) - E(4,2)\Big)t_{(31)}^2 + \\
& +\frac{3g^2}{16}\Big(1 - 2E(2,2) - 2E(4,2) + E(6,4)\Big)t_{(32)}^2 + \\
& +\frac{\sqrt{3}g^2}{16}\Big(E(-2,0) - 6E(2,2) + E(6,4)\Big)t_{(21)}t_{(01)} + \\
& +\frac{\sqrt{3}g^2}{16}\Big(E(2,0) - 6E(4,2) + E(6,4)\Big)t_{(11)}t_{(31)} \,.
\end{aligned}
\tag{3.15}
$$

Note that up to this order the scalar action is diagonal in $t_{(10)}$ and $t_{(32)}$ and there are two 2×2 blocks in $t_{(11)}/t_{(31)}$ and $t_{(21)}/t_{(01)}$, respectively.

The potential in (3.13) and (3.15) involves all eight SO(3)-singlet fields, but is still invariant under the $\mathrm{SL}(2,\mathbb{R})_\tau$ and an $O(2)$ subgroup of the gauge group. The

[6] An interested reader may enjoy it at `http://citusc.usc.edu/~pilch/Papers/g2potexp.out`.

former symmetry is given by the transformations (3.4) acting on the $\mathrm{SL}(2,\mathbb{R})$ index α of the 27-bein. This is a manifest local symmetry of the potential of the full $\mathcal{N}=8$ theory [51]. However, its action on the SO(3)-singlet fields in our solvable parameterization is quite complicated as might be inferred from the explicit form of the $\mathrm{SL}(2,\mathbb{R})_\tau$ generators, given by

$$
\begin{aligned}
T_+ &= -\frac{\sqrt{3}}{2}(X_{(11)} + X_{(-1-1)}) + \frac{1}{2}(X_{(31)} + X_{(-3-1)})\,, \\
T_- &= \frac{\sqrt{3}}{2}(X_{(21)} + X_{(-2-1)}) - \frac{1}{2}(X_{(01)} + X_{(0-1)})\,, \\
T_3 &= -\frac{\sqrt{3}}{2}(X_{(10)} - X_{(-10)}) + \frac{1}{2}(X_{(32)} - X_{(-3-2)})\,,
\end{aligned}
\tag{3.16}
$$

which act on the scalar coset, $\mathcal{Q}_0$, by right multiplication of (3.10). The generator of the $O(2)$ symmetry is given in (3.17) below.

If one is interested in studying the potential alone, it is more advantageous to use another parameterization of $\mathcal{Q}_0$ and gauge fix all symmetries. This was done in [27], where the potential for the SO(3)-singlet fields was shown to depend on m and σ and two other fields, which together parametrized a quotient of $\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SU}(2)$ by an $O(2)$ action. However, one must remember that the above local symmetries of the potential are *not* local symmetries of the full scalar action and thus a careful treatment of the dynamics of the SO(3)-singlet sector requires that we consider all eight fields. For this purpose the solvable parameterization employed here is quite convenient.

An action for some of the SO(3)-invariant scalars of the gauged $\mathcal{N}=8$ theory was also derived and studied in [20]. The authors restrict to a 6-dimensional scalar subspace, and their scalar potential depends only on 2 of the 6 scalars, not including the dilaton nor any field mixing with it. A direct comparison of our results with [20] is rather difficult because complicated field redefinitions are required (although we have reproduced the truncated action in their parameterization). Nevertheless we believe that their results are incorrect because there is no natural 6-dimensional subspace of the coset $G_{2(2)}/\,\mathrm{SO}(4)$ nor any symmetry to enforce such a truncation. A more precise criticism is that their kinetic action for the dilaton gives a fluctuation equation (about the GPPZ flow) which is not hypergeometric, as opposed to those of all other SO(3)-singlet fluctuations we study. As shown in section 6 below, the dilaton and axion emerge precisely from the mixed 2×2 sectors of the quadratic potential (3.15). This is a consequence of the action and equations of motion for these fields, and doesn't depend on the form of the background $\{m(r),\sigma(r)\}$ or the presence of supersymmetry. For these reasons the 6-dimensional truncation in [20] appears to be inconsistent and the application to chiral symmetry breaking should be reexamined.

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3.2 The vector field and the vector/scalar coupling

The commutant of SO(3) in the gauge group SO(6) consists of a single U(1) generated by the transformations with $\Lambda^1{}_4 = \Lambda^2{}_5 = \Lambda^3{}_6$ and the corresponding generator in the Cartan basis given by

$$T_R = -\frac{\sqrt{3}}{2}\left(X_{(10)} - X_{(-10)}\right) - \frac{3}{2}\left(X_{(32)} - X_{(-3-2)}\right). \tag{3.17}$$

We will denote this subgroup by $U(1)_R$ to emphasize that it corresponds to the truncation of the R-symmetry group on the field theory side. This symmetry is, however, broken by the background fields m and σ and as a result the corresponding vector field, B_μ, develops an m- and σ-dependent mass-term. A tedious expansion of the $\mathcal{N}=8$ $D=5$ action to the quadratic order in B_μ yields

$$\mathcal{L}(B_\mu; m,\sigma) = -\frac{3}{4}F_{\mu\nu}F^{\mu\nu} + \frac{3}{16}g^2\left(3\cosh(4\sigma) + \cosh\left(\frac{4m}{\sqrt{3}}\right) - 4\right)B_\mu B^\mu . \tag{3.18}$$

In deriving (3.18) we have used that there are no SO(3)-singlet antisymmetric tensor fields that might mix with B_μ. One can use (2.13) and the flow equations (2.7), (2.10) to show that the vector mass is $m_B^2 = -2A''(r)$. We will show in the next section that this result is quite generic.

The coupling between the vector and the scalar fields up to the quadratic order in the fluctuations is given by

$$\begin{aligned}\mathcal{L}^{(2)}(B_\mu, t_\alpha; m,\sigma) = B_\mu \Bigg(& 3(\partial^\mu\sigma)t_{(32)} - (\partial^\mu m)t_{(10)} + \frac{\sqrt{3}}{4}(1-e^{-4m/\sqrt{3}})\partial^\mu t_{(10)} + \\ & + \frac{3}{4}(1-e^{4\sigma})\partial^\mu t_{(32)}\Bigg) ,\end{aligned} \tag{3.19}$$

which is in agreement with the usual gauging of a nonlinear σ-model as we show in more detail in section 6.

3.3 The fermion fields

The SO(3) truncation of the fermion sector yields one symplectic Majorana pair of spin-3/2 fields, $\tilde{\psi}^a_\mu$, $a=1,2$, and two pairs of spin-1/2 fields, $\tilde{\chi}^a$, $a=1,\ldots,4$. This follows from the counting of SO(3) singlets in the branching of **8** and **48** of USp(8). We use formulae in Appendix A of [51] for the embedding of SO(3) into USp(8) (with the SO(7) gamma matrices given in Appendix C of [9]) to determine those SO(3)-singlet fields in terms of the original fields of the $\mathcal{N}=8$ theory:

$$\tilde{\psi}^1_\mu = \psi^3_\mu , \qquad \tilde{\psi}^2_\mu = \psi^7_\mu , \tag{3.20}$$

which mirrors the truncation of the supersymmetry parameter and

$$
\begin{aligned}
\tilde{\chi}^1 &= \chi^{124}\,, \\
\tilde{\chi}^2 &= \chi^{128} = -\chi^{146} = \chi^{245}\,, \\
\tilde{\chi}^3 &= \chi^{168} = -\chi^{258} = \chi^{456}\,, \\
\tilde{\chi}^4 &= \chi^{568}\,.
\end{aligned} \tag{3.21}
$$

In (3.21) we have used that $\chi^{abc} = \chi^{[abc]|}$ are antisymmetric and symplectic traceless USp(8) tensors and listed the independent components only.

The reduction of the fermion action is now rather easy and we find that further redefinitions of the spin-1/2 fields are required to bring it into canonical form. Let us define

$$
\begin{aligned}
\xi^1 &= \frac{\sqrt{3}}{2}(\tilde{\chi}^2 + \tilde{\chi}^4)\,, \qquad & \xi^2 &= \frac{\sqrt{3}}{2}(\tilde{\chi}^1 + \tilde{\chi}^3)\,, \\
\xi^3 &= -\frac{1}{2}(3\tilde{\chi}^2 - \tilde{\chi}^4)\,, \qquad & \xi^4 &= \frac{1}{2}(\tilde{\chi}^1 - 3\tilde{\chi}^3)\,.
\end{aligned} \tag{3.22}
$$

In terms of those fields the fermion kinetic action becomes

$$
\begin{aligned}
\mathcal{L}_K^{(2)}(\psi_\mu^a, \xi^a) = &-\frac{i}{2}\left(\bar{\psi}_\mu^1 \gamma^{\mu\nu\rho} D_\nu \psi_\rho^2 - \bar{\psi}_\mu^2 \gamma^{\mu\nu\rho} D_\nu \psi_\rho^1\right) \\
&-\frac{i}{2}\left(\bar{\xi}^1 \gamma^\mu D_\mu \xi^2 - \bar{\xi}^2 \gamma^\mu D_\mu \xi^1 + \bar{\xi}^3 \gamma^\mu D_\mu \xi^4 - \bar{\xi}^4 \gamma^\mu D_\mu \xi^3\right)\,,
\end{aligned} \tag{3.23}
$$

where we dropped the tilde over the spin-3/2 field. The truncation of the mass terms is more complicated as it requires evaluating the USp(8)-tensors of the $\mathcal{N} = 8$ theory. We find

$$
\begin{aligned}
\mathcal{L}_M^{(2)}(\psi_\mu^a, \xi^a) = &-\frac{ig}{4} W(m,\sigma)(\bar{\psi}_\mu^1 \gamma^{\mu\nu} \psi_\nu^1 + \bar{\psi}_\mu^2 \gamma^{\mu\nu} \psi_\nu^2) - \\
&-\frac{ig}{16}\left(\cosh\left(\frac{2m}{\sqrt{3}}\right) - 3\cosh(2\sigma)\right)\left(\bar{\xi}^1 \xi^1 + \bar{\xi}^2 \xi^2 - 3\bar{\xi}^3 \xi^3 - 3\bar{\xi}^4 \xi^4\right) + \\
&+\frac{1}{\sqrt{2}} m'(r)\left(\bar{\psi}_\mu^1 \gamma^r \gamma^\mu \xi^1 + \bar{\psi}_\mu^2 \gamma^r \gamma^\mu \xi^2\right) + \\
&+\frac{1}{\sqrt{2}} \sigma'(r)\left(\bar{\psi}_\mu^1 \gamma^r \gamma^\mu \xi^3 + \bar{\psi}_\mu^2 \gamma^r \gamma^\mu \xi^4\right) - \\
&-\frac{\sqrt{3}g}{4\sqrt{2}} \sinh\left(\frac{2m}{\sqrt{3}}\right)\left(\bar{\xi}^2 \gamma^\mu \psi_\mu^1 - \bar{\xi}^1 \gamma^\mu \psi_\mu^2\right) - \\
&-\frac{\sqrt{3}g}{4\sqrt{2}} \sinh(2\sigma)\left(\bar{\xi}^4 \gamma^\mu \psi_\mu^1 - \bar{\xi}^3 \gamma^\mu \psi_\mu^2\right)\,,
\end{aligned} \tag{3.24}
$$

where $W(m, \sigma)$ is the superpotential (2.13). This may be further simplified using the flow equations (2.7), (2.10), which we will do in section 7.

3.4 The supersymmetry transformations

We will now verify that to the linear order we have indeed obtained a gauged $\mathcal{N}=2$ supergravity coupled to two hypermultiplets. For the supergravity multiplet we find:

$$
\begin{aligned}
\delta e_\mu^m &= -i(\bar\epsilon^1\gamma^m\psi_\mu^2 - \bar\epsilon^2\gamma^m\psi_\mu^1)\,,\\
\delta\psi_\mu^1 &= D_\mu\epsilon^1 - \frac{1}{6}gW(m,\sigma)\gamma_\mu\epsilon^2 - gB_\mu W(m,\sigma)\epsilon^2 + \frac{1}{4}F_{\nu\rho}(\gamma^{\nu\rho}\gamma_\mu + 2\gamma^\nu\delta_\mu^\rho)\epsilon^1 +\\
&\quad + \frac{ig}{2}\left(E(0,1)\partial_\mu t_{(01)} - \sqrt{3}E(2,1)\partial_\mu t_{(21)}\right)\epsilon^1 -\\
&\quad - \frac{g}{2}\left(\sqrt{3}E(1,0)\partial_\mu t_{(10)} + \sqrt{3}iE(1,1)\partial_\mu t_{(11)} - iE(3,1)\partial_\mu t_{(31)} + E(3,2)\partial_\mu t_{(32)}\right)\epsilon^2,\\
\delta\psi_\mu^2 &= D_\mu\epsilon^2 + \frac{1}{6}gW(m,\sigma)\gamma_\mu\epsilon^1 + gB_\mu W(m,\sigma)\epsilon^1 + \frac{1}{4}F_{\nu\rho}(\gamma^{\nu\rho}\gamma_\mu + 2\gamma^\nu\delta_\mu^\rho)\epsilon^2 +\\
&\quad + \frac{g}{2}\left(\sqrt{3}E(1,0)\partial_\mu t_{(10)} - \sqrt{3}iE(1,1)\partial_\mu t_{(11)} + iE(3,1)\partial_\mu t_{(31)} + E(3,2)\partial_\mu t_{(32)}\right)\epsilon^1 -\\
&\quad - \frac{ig}{2}\left(E(0,1)\partial_\mu t_{(01)} - \sqrt{3}E(2,1)\partial_\mu t_{(21)}\right)\epsilon^2\,,\\
\delta B_\mu &= -\frac{i}{4}\,(\bar\epsilon^1\psi_\mu^2 - \bar\epsilon^2\psi_\mu^1)\,.
\end{aligned}
\tag{3.25}
$$

To present the supersymmetry transformations for the hypermultiplets, let us define rescaled variation of t_α, cf. (3.12),

$$
\tilde\delta t_\alpha = e^{\alpha(m,\sigma)}\delta t_\alpha\,,
\tag{3.26}
$$

in terms of which the result has a compact form:

$$
\begin{aligned}
\delta m &= -\frac{1}{\sqrt{2}}(\bar\epsilon^1\xi^1 + \bar\epsilon^2\xi^2)\,,\\
\tilde\delta t_{(10)} &= -\bar\epsilon^1\xi^2 + \bar\epsilon^2\xi^1\,,\\
\frac{\sqrt{3}}{2}\tilde\delta t_{(01)} + \frac{1}{2}\tilde\delta t_{(21)} &= i(\bar\epsilon^1\xi^1 - \bar\epsilon^2\xi^2)\,,\\
\frac{\sqrt{3}}{2}\tilde\delta t_{(31)} + \frac{1}{2}\tilde\delta t_{(11)} &= -i(\bar\epsilon^1\xi^2 + \bar\epsilon^2\xi^1)\,,
\end{aligned}
\tag{3.27}
$$

and

$$
\begin{aligned}
\delta\sigma &= -\frac{1}{\sqrt{2}}(\bar\epsilon^1\xi^3 + \bar\epsilon^2\xi^4)\,,\\
\tilde\delta t_{(32)} &= (\bar\epsilon^1\xi^4 - \bar\epsilon^2\xi^3)\,,\\
\frac{1}{2}\tilde\delta t_{(01)} - \frac{\sqrt{3}}{2}\tilde\delta t_{(21)} &= i(\bar\epsilon^1\xi^3 - \bar\epsilon^2\xi^4)\,,\\
-\frac{1}{2}\tilde\delta t_{(31)} + \frac{\sqrt{3}}{2}\tilde\delta t_{(11)} &= -i(\bar\epsilon^1\xi^4 + \bar\epsilon^2\xi^3)\,.
\end{aligned}
\tag{3.28}
$$

The particular combinations of the scalar fields we have displayed here turn out to be crucial for the diagonalization of the field equations in section 6.

Similarly, for the variation of the spin-1/2 fields let us use the notation

$$\tilde{\not\partial} t_\alpha = e^{\alpha(m,\sigma)}\ \not\partial t_\alpha\,. \tag{3.29}$$

Then

$$\begin{aligned}
\delta\xi^1 = {} & \frac{i}{\sqrt{2}}\ \not\partial m\epsilon^2 - \frac{1}{4}\left(\sqrt{3}\tilde{\not\partial} t_{(01)} + \tilde{\not\partial} t_{(21)}\right)\epsilon^2 + \frac{i}{2}\tilde{\not\partial} t_{(10)}\,\epsilon^1 - \\
& - \frac{1}{4}\left(\sqrt{3}\tilde{\not\partial} t_{(31)} + \tilde{\not\partial} t_{(11)}\right)\epsilon^1 - \frac{ig\sqrt{3}}{4\sqrt{2}}\ \sinh\left(\frac{2m}{\sqrt{3}}\right)\epsilon^1 + \\
& + \frac{ig\sqrt{3}}{2\sqrt{2}}\ \not B\ \sinh\left(\frac{2m}{\sqrt{3}}\right)\epsilon^1 + \frac{g\sqrt{3}}{16}(3E(-3,-1) + E(1,1))\, t_{(01)}\,\epsilon^1 - \\
& - \frac{g}{16}(3E(-2,-1) + 3E(0,1) - 2E(2,1))\, t_{(11)}\,\epsilon^2 + \\
& + \frac{g}{16}(3E(-1,-1) - 2E(1,1) + 3E(3,1))\, t_{(21)}\,\epsilon^1 - \\
& - \frac{g\sqrt{3}}{16}(3E(0,-1) + E(2,1))\, t_{(31)}\,\epsilon^2 + \frac{ig}{4}\, t_{(10)}\,\epsilon^2
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
\delta\xi^2 = {} & -\frac{i}{\sqrt{2}}\ \not\partial m\,\epsilon^1 - \frac{1}{4}\left(\sqrt{3}\tilde{\not\partial} t_{(01)} + \tilde{\not\partial} t_{(21)}\right)\epsilon^1 + \frac{i}{2}\tilde{\not\partial} t_{(10)}\,\epsilon^2 + \\
& + \frac{1}{4}\left(\sqrt{3}\tilde{\not\partial} t_{(31)} + \tilde{\not\partial} t_{(11)}\right)\epsilon^2 - \frac{ig\sqrt{3}}{4\sqrt{2}}\ \sinh\left(\frac{2m}{\sqrt{3}}\right)\epsilon^2 + \\
& + \frac{ig\sqrt{3}}{2\sqrt{2}}\ \not B\ \sinh\left(\frac{2m}{\sqrt{3}}\right)\epsilon^2 - \frac{g\sqrt{3}}{16}(3E(-3,-1) + E(1,1))\, t_{(01)}\,\epsilon^2 - \\
& - \frac{g}{16}(3E(-2,-1) + 3E(0,1) - 2E(2,1))\, t_{(11)}\,\epsilon^1 - \\
& - \frac{g}{16}(3E(-1,-1) - 2E(1,1) + 3E(3,1))\, t_{(21)}\,\epsilon^2 - \\
& - \frac{g\sqrt{3}}{16}(3E(0,-1) + E(2,1))\, t_{(31)}\,\epsilon^1 - \frac{ig}{4}\, t_{(10)}\,\epsilon^1\,.
\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}
\delta\xi^3 = {} & \frac{i}{\sqrt{2}}\ \not\partial\sigma\,\epsilon^2 - \frac{1}{4}\left(\tilde{\not\partial} t_{(01)} - \sqrt{3}\tilde{\not\partial} t_{(21)}\right)\epsilon^2 - \frac{i}{2}\tilde{\not\partial} t_{(32)}\,\epsilon^1 + \\
& + \frac{1}{4}\left(\tilde{\not\partial} t_{(31)} - \sqrt{3}\tilde{\not\partial} t_{(11)}\right)\epsilon^2 - \frac{3ig}{4\sqrt{2}}\ \sinh(2\sigma)\,\epsilon^1 + \\
& + \frac{3ig}{2\sqrt{2}}\ \not B\ \sinh(2\sigma)\,\epsilon^1 - \frac{3g}{16}(E(-3,-1) - E(1,1))\, t_{(01)}\,\epsilon^1 - \\
& - \frac{g\sqrt{3}}{16}(-3E(-2,-1) + E(0,1) + 2E(2,1))\, t_{(11)}\,\epsilon^2 - \\
& - \frac{g\sqrt{3}}{16}(-3E(-1,-1) + 2E(1,1) + E(3,1))\, t_{(21)}\,\epsilon^1 - \\
& - \frac{3g}{16}(E(0,-1) - E(2,1))\, t_{(31)}\,\epsilon^2 - \frac{3ig}{4}\, t_{(32)}\,\epsilon^2\,,
\end{aligned} \tag{3.32}$$

$$
\begin{aligned}
\delta\xi^4 = &-\frac{i}{\sqrt{2}}\,\partial\!\!\!/\sigma\,\epsilon^1 - \frac{1}{4}\left(\tilde{\partial\!\!\!/}t_{(01)} - \sqrt{3}\tilde{\partial\!\!\!/}t_{(21)}\right)\epsilon^1 - \frac{i}{2}\tilde{\partial\!\!\!/}t_{(32)}\,\epsilon^2 - \\
&-\frac{1}{4}\left(\tilde{\partial\!\!\!/}t_{(31)} - \sqrt{3}\tilde{\partial\!\!\!/}t_{(11)}\right)\epsilon^2 - \frac{3ig}{4\sqrt{2}}\sinh(2\sigma)\,\epsilon^2 + \\
&+\frac{3ig}{2\sqrt{2}}\,B\!\!\!/\,\sinh(2\sigma)\,\epsilon^2 + \frac{3g}{16}(E(-3,-1) - E(1,1))t_{(01)}\,\epsilon^2 - \\
&-\frac{g\sqrt{3}}{16}(-3E(-2,-1) + E(0,1) + 2E(2,1))\,t_{(11)}\,\epsilon^1 - \\
&-\frac{g\sqrt{3}}{16}(3E(-1,-1) - 2E(1,1) - E(3,1))\,t_{(21)}\,\epsilon^2 - \\
&-\frac{3g}{16}(E(0,-1) - E(2,1))\,t_{(31)}\,\epsilon^1 + \frac{3ig}{4}\,t_{(32)}\,\epsilon^1\,,
\end{aligned}
\tag{3.33}
$$

When the background satisfies the flow equations (2.7), (2.10) the zeroth-order terms in $\delta\psi_\mu$ and $\delta\xi$ will vanish for ϵ a Killing spinor, as given by (2.8).

4. Physical modes of the gravity multiplet

In this section we consider the physical (i.e. transverse, traceless) modes of the metric fluctuation $h_{\mu\nu} \equiv e^{-2A}\,\hat{\delta}g_{\mu\nu}$, gravitino ψ^a_μ, and U$(1)_R$ photon B_μ, which are dual to the superconformal current multiplet containing the stress tensor, supercurrent and R-current. These TT modes are non-vanishing only for transverse values of the indices, i.e. $\mu,\nu \to i,j$, and they will decouple from other modes and from lower spin fluctuations; we denote them $\{\hat{h}_{ij}, \hat{\psi}_i, \hat{B}_i\}$.

In the GPPZ background, these modes obey the linear equations of motion:

$$R^{(1)}_{ij}(\hat{h}) + \frac{4}{3}V(m(r))\,\hat{h}_{ij} = 0\,, \tag{4.1}$$

$$\gamma^{\mu\nu\rho}\hat{D}_\nu\hat{\psi}_\rho = 0\,, \tag{4.2}$$

$$D^\mu\hat{F}_{\mu\nu} - 2A''(r)\hat{B}_\nu = 0\,, \tag{4.3}$$

where we combined the symplectic Majorana gravitini into a complex field $\psi_\rho \equiv \psi^1_\rho + i\psi^2_\rho$, and

$$\hat{D}_\nu \equiv D_\nu - \frac{ig}{6}W\gamma_\nu\,. \tag{4.4}$$

We will not write the linearized Ricci operator $R^{(1)}_{ij}$ in detail (see for example [59]), since it is by now well-known (as derived in [12, 59] for a general domain wall and earlier in a special case [60]) that the TT modes of $\hat{h}_{ij}$ obey the same equation as that of a massless scalar,

$$D^\mu\partial_\mu f(x^i,r) = \left[-\partial_r^2 - 4A'\partial_r + e^{-2A}\Box\right] f(x^i,r) = 0\,, \qquad \hat{h}_{ij} \equiv v_i v_j f(x^i,r)\,, \tag{4.5}$$

with v_i independent of r and transverse. The TT constraints then require that the v_i are constructed from three independent polarization vectors (as described in detail in (21) of [43]). In this way one can build the five independent physical modes of $\hat{h}_{ij}$. The solution for the GPPZ flow was found to be [38, 43]:

$$f = (1-u)^2 \, F\left(2 + \frac{pL}{2}, 2 - \frac{pL}{2}; 2; u\right) , \tag{4.6}$$

where the radial variable u is defined in (2.15) and $F(a,b,c;z)$ denotes a standard hypergeometric function [61]. The associated spectrum contains discrete states with momenta $p^2L^2 = 4(n+2)^2$, in agreement with (1.1). It should be noted the SO(3)-singlet supergravity theory does not contain a massless scalar field; f is just a convenient auxiliary quantity.

Since $\hat{\psi}_\mu$ and $\hat{B}_\mu$ are SUSY partners of $\hat{h}_{ij}$, we expect that they can also be expressed in terms of the massless scalar f, and it is our purpose to show how to do this. We use supersymmetry transformation rules to relate solutions of the equations of motion (4.1)–(4.3). The necessary parts of the transformations (3.25), rewritten in terms of complex spinors, are

$$\delta e_\mu^{\hat{m}} = \bar{\epsilon}\gamma^{\hat{m}}\psi_\mu \, , \tag{4.7}$$

$$\delta\psi_\mu = \hat{D}_\mu \epsilon \, , \tag{4.8}$$

$$\delta B_\mu = \frac{1}{4}\bar{\epsilon}\,\psi_\mu \, , \tag{4.9}$$

where ϵ is the Killing spinor of (2.8).

Let $\hat{h}_{ij}(x^i, r)$ be any solution of (4.1). The corresponding fluctuation of the frame can be chosen as

$$\hat{\delta}\hat{e}_j^{\hat{k}} = \frac{1}{2} e^{A(r)} \eta^{\hat{k}i} \hat{h}_{ij}(x^k, r) \, , \tag{4.10}$$

where all other components vanish and $\eta^{\hat{k}i}$ is the 4×4 Minkowski metric. Let $\hat{\delta}\omega_{\mu\hat{k}\hat{l}}$ denote the corresponding fluctuation of the spin connection. It is then guaranteed that

$$\hat{\psi}_\mu = \left(\partial_\mu + \frac{1}{4}\,\hat{\delta}\omega_{\mu\hat{k}\hat{l}}\gamma^{\hat{k}\hat{l}} + \frac{i}{2}A'(r)\gamma_{\hat{k}}\hat{\delta}e_\mu^{\hat{k}}\right)\epsilon \, , \tag{4.11}$$

is a solution of (4.2). It is now straightforward to use the specific form of $\hat{\delta}\omega_{\mu\hat{k}\hat{l}}$ and show that the radial component $\hat{\psi}_5$ vanishes, and that all terms in which $\hat{h}_{ij}$ is not differentiated cancel in $\hat{\psi}_i$. One can then use the transverse traceless property of $\hat{h}_{ij}$ to bring $\hat{\psi}_i$ to the form

$$\hat{\psi}_i = \frac{1}{4}\, e^{2A(r)}\gamma^j(\gamma^\mu \partial_\mu \hat{h}_{ij})\epsilon \, . \tag{4.12}$$

Despite appearances this form is covariant since the factor $e^{2A(r)}$ can be moved to the right of the derivative. One then finds that ∂_μ is replaced by the covariant D_μ acting on the tensor fluctuation $\hat{\delta}g_{ij} = e^{2A(r)}\hat{h}_{ij}$. It is easy to see that $\gamma^i\hat{\psi}_i$ and $g^{ij}D_i\hat{\psi}_j$ vanish, so these gravitino modes are transverse traceless.

The final step is to express $\hat{\psi}_i$ in terms of the scalar f. To do this we simply substitute $\hat{h}_{ij} = v_i v_j f$ in (4.12) to obtain

$$\hat{\psi}_i = -\frac{1}{4}\, e^{2A(r)} v_i (\gamma^\mu \partial_\mu f)(\gamma^j v_j)\epsilon\,. \tag{4.13}$$

It is convenient to define a new spinor $\tilde{\epsilon} = (\gamma^j v_j)\epsilon$. This might be called an anti-Killing spinor, since the form of $\tilde{\epsilon}$ is the same as (2.8) but with opposite chirality of the constant spinor $\eta^{(0)}$, and consequently $(D_\mu + \frac{ig}{6} W\gamma_\mu)\,\tilde{\epsilon} = 0$. Dropping an irrelevant constant we can then write

$$\hat{\psi}_i = e^{2A(r)} v_i\, (\gamma^\mu \partial_\mu f)\, \tilde{\epsilon}\,. \tag{4.14}$$

We have verified that this form is indeed a solution of (4.2). The next issue is to count the linearly independent modes of the form (4.14). This requires detailed analysis in which specific polarization vectors are paired with choices of η to create modes of overall half-integer helicity. There are a total of 4 independent modes, as expected.[7]

We proceed now to treat the graviphoton field B_μ in the same fashion. Substituting $\hat{\psi}_i$ from (4.14) into (4.9), we find

$$\hat{B}_i \sim e^{2A(r)}\, v_i\, \bar{\epsilon}(\gamma^\mu \partial_\mu f)\, \tilde{\epsilon}\,, \tag{4.15}$$

and SUSY guarantees that this is a solution of (4.3). Since Killing and anti-Killing spinors have opposite chirality this simplifies to the form

$$\hat{B}_i = e^{2A(r)}\, v_i\, \partial_r f(x^i, r)\,. \tag{4.16}$$

It is easy to verify that, if $f(x^i, r)$ satisfies (4.5), then $\hat{B}_i$ satisfies the massive vector equation of motion (4.3) which can be written in explicit form as

$$-(\partial_r + 2A'(r))\partial_r \hat{B}_i + e^{-2A(r)} \Box \hat{B}_i - 2A''(r)\hat{B}_i = 0\,. \tag{4.17}$$

This completes our presentation of the TT modes of the gravity multiplet in terms of the auxiliary scalar f.

The equations (4.1), (4.2) and (4.3) have been extracted from those of the full gauged $\mathcal{N} = 8$ theory linearized around the GPPZ flow, as can be seen explicitly from (3.18) and (3.23). Further, equation (4.1) is known to be universal for any RG flow geometry. The gravitino equation (4.2) obtains for any reduction of the $\mathcal{N} = 8$ theory, since the only other term in the gravitino field equation is proportional to γ^μ and cannot contribute [51]; it is quite generic, since it consists merely of the canonical Rarita-Schwinger kinetic term and a mass term.

Equation (4.3) is more particular; we will find a different equation for the Coulomb branch graviphoton in section 8. One may show, however, that given

[7] It may appear that the form (4.13) contains more modes. However, one must take into account the fact that $\gamma^j v_j$ has zero modes when v_j is a circular polarization vector.

the assumption of a graviphoton with canonical kinetic terms and some mass m_B^2, the transformation (4.9) and consistency with the gravitino equation (4.2) require $m_B^2 = -2A''$. As we will discuss, flows with active scalars in hypermultiplets must have canonical vector kinetic terms, and are associated with backgrounds with broken $\mathrm{U}(1)_R$. Thus it seems likely that the form (4.3) is generic for flows where R-symmetry is broken, and the gauge field acquires a mass. One should note that $m_B^2 = -2A''(r)$ is non-negative, as a consequence of the holographic c-theorem [8, 9].

In the Coulomb case, R-symmetry is preserved and the graviphoton remains massless. However the active scalar, which sits in a vector multiplet, produces a non-canonical kinetic term for B_μ. We will show in section 8, however, that the graviphoton equation of motion can be transformed into (4.17), including the same mass. For all these reasons the vector mass $m_B^2 = -2A''$ appears to be generic in RG flows.

5. The anomaly multiplet

As we have seen in the previous section, the transverse and traceless modes of the gravity multiplet do not mix with other fields, and can be collectively described in terms of an auxiliary free massless scalar f. The remaining modes in the gravity multiplet vanish or can be gauged away in an anti-de Sitter background. In an RG flow background, however, the story is not so simple, as the profile of the active scalar ϕ couples them to other fields.

The mixing of the graviton trace and the fluctuations of the active scalar(s) was first discussed in [43]. The equations of motion governing the coupled system are quite general, regardless of the character of the background flow. The system was examined in an axial gauge, and for the case of a single active scalar, was reduced to an uncoupled third-order equation. This was solved for the GPPZ and Coulomb branch flows. Translating the solutions into sensible correlation functions, however, proved difficult.

The problem was taken up by Arutyunov, Frolov and Theisen (AFT) [44], who employed a different gauge and a different prescription for correlation functions. They obtained a solution for the GPPZ flow, which we will show shortly to be gauge equivalent to the solution of [43]. AFT did not solve the corresponding equation for the Coulomb branch flow, but in fact the gauge transform of the solution from [43] satisfies their equation, as we shall describe. Thus the solutions are equivalent.

In the next several sections, we shall illustrate how the coupling of the gravity trace to the active scalar in the GPPZ flow generalizes to a coupling between the "trace" of the gravity multiplet and the active hypermultiplet. The trace of the gravity multiplet contains the graviton trace, gravitino γ-trace and the longitudinal graviphoton, which are dual in the field theory to the trace of the stress tensor, the γ-trace of the supercurrent and the divergence of the R-current. These operators constitute a chiral multiplet called the anomaly multiplet, which vanishes when conformal invariance is unbroken. In the GPPZ flow background, $\gamma^\mu\psi_\mu$ couples to the spin-1/2 fields in the active hypermultiplet, while the phase associated to the active scalar Higgses the graviphoton, corresponding to the breaking of the R-symmetry. Thus the coupling we uncover between the traces of the gravity multiplet and the active hypermultiplet agrees perfectly with field theory expectations: we can identify the multiplet of the active scalar as the "anomaly hypermultiplet".

We shall examine, in turn, the graviton trace/active scalar system, the vector/scalar system, and the fermion sector of the GPPZ flow. We shall see that all the anomaly multiplet fields (as well as the traces of the gravity multiplet which mix with them) have a common spectrum of states distinct from the transverse traceless gravity fields, while the uncoupled lagrangian multiplet is characterized by a third spectrum.

Interestingly, the situation is not quite the same in the Coulomb branch flow. We will examine this case in section 8.

5.1 Axial and AFT gauges for graviton trace/active scalar sector

The coupled $h^\mu_\mu/\tilde\phi$ system was considered in two different gauges in [43] and [44]. Here we will demonstrate the gauge equivalence of the solutions for the GPPZ flow, and show that the gauge transform of the axial gauge Coulomb branch solution solves the AFT fluctuation equation. Finally we will demonstrate that the effective scalar s defined by AFT can be interpreted as the active scalar $\tilde\phi$ itself in a third gauge. We calculate the 2-point functions for the active scalar in both flows; the result for the GPPZ flow agrees with [44] up to the lack of a massless pole, while the result for the Coulomb-branch flow displays the usual mass gap and continuum.

The most general form of the metric and scalar we use will be

$$\begin{aligned} ds^2 &= e^{2A(r)}\left(\eta_{ij}+h_{ij}(r,x)\right)dx^i dx^j + (-1+h_{55}(r,x))dr^2\,, \\ \phi_{\rm tot} &= \phi(r)+\tilde\phi(r,x)\,, \end{aligned} \tag{5.1}$$

where we have already gauged away possible h_{i5} components. Since the five TT modes of h_{ij} have been discussed in section 4, and three additional longitudinal modes can be gauged away using (5.10) below (see section 2.2 of [43]), it is sufficient to restrict to the trace components of h_{ij},

$$h_{ij}(r,x) = e^{ipx}\left(\frac{1}{4}\,h(r,p)\,\eta_{ij} + p_i\,p_j\,H(r,p)\right). \tag{5.2}$$

The holographic β-function [14, 36, 38] of the operator $\mathcal{O}_\phi$ dual to ϕ is defined using $a \equiv e^A$ as the scale in the 4D theory:

$$\beta_\phi = a\frac{d}{da}\phi = \frac{\phi'(r)}{A'(r)} = -\frac{3}{2W}\frac{\partial W}{\partial\phi}\,. \tag{5.3}$$

This differs by an overall sign from the β-function used by AFT.

The fields h, H, h_{55} and $\tilde{\phi}$ all mix. However, residual gauge freedom can be used to eliminate one of them. In [43] an axial gauge choice,

$$h_{i5}^{\rm axial} = h_{55}^{\rm axial} = 0\,, \tag{5.4}$$

was made, while AFT employed the gauge

$$h_{i5}^{AFT} = \tilde{\phi}^{AFT} = 0\,. \tag{5.5}$$

Let us consider residual diffeomorphisms defined by a vector field $v_\mu(r,x)$. In both gauges, we must require

$$\delta h_{i5} = 0 = \partial_i v_5 + \partial_5 v_i - 2A' v_i = 0\,, \tag{5.6}$$

which determines v_i in terms of v_5,

$$v_i(r,x) = -e^{2A(r)} \int^r dr' e^{-2A(r')} \partial_i v_5(r',x)\,. \tag{5.7}$$

In axial gauge, we must also enforce

$$\delta h_{55}^{\rm axial} = 0 = \partial_r v_5(r,x)\,. \tag{5.8}$$

While in AFT gauge, instead we have

$$\delta\tilde{\phi}_{AFT} = 0 = v^\mu \partial_\mu \phi(r) = -v_5 \phi'(r)\,, \tag{5.9}$$

where $\phi(r)$ is the background scalar profile. In both gauges, there are residual transformations of the same form as 4D diffeomorphisms:

$$\begin{aligned} v_i &= e^{2A(r)}\, w_i(x), \qquad v_5 = 0\,, \\ \delta h_{ij} &= \partial_i\, w_j(x) + \partial_j\, w_i(x)\,. \end{aligned} \tag{5.10}$$

These certainly satisfy (5.6) and (5.8) or (5.9), and we may think of them as coming from the lower limit of integration in (5.7).

In axial gauge there are also less trivial residual gauge transformations which are the linearization of the subgroup of bulk diffeomorphisms which induce Weyl transformations of the boundary metric. They have been studied in the context[8] of the AdS/CFT correspondence in [62], although their identification is much older [63, 64]. They are generated by arbitrary $v_5 = v_5(x)$, independent of r, with v_i determined by (5.7). These transformations give a pure gauge solution [43] of the fluctuation equations in axial gauge, namely, if $v_5(x) = e^{ipx} E(p)$,

$$\delta h = -8A'(r)\, E(p)\,, \qquad \delta H' = 2\, e^{-2A(r)}\, E(p)\,, \qquad \delta\tilde{\phi} = -\phi'(r)\, E(p)\,. \tag{5.11}$$

[8]DZF thanks Kostas Skenderis for useful discussions of this point.

However, because v_5 appears without derivative in (5.9), there is no corresponding residual transformation in AFT gauge, which is thus a more complete gauge fix.

The fluctuation equations in axial gauge have the spurious solution (5.11) in addition to physical solutions, and this appears to be the reason why these equations are more complicated and can at best be reduced to an uncoupled third order equation for $\tilde{\phi}$. In *AFT* gauge there is no obstruction to a second order equation.

The solutions found by [43] and [44] are nonetheless equivalent. Consider a solution $\{h, H, \tilde{\phi}\}$ in axial gauge. Let us transform to AFT gauge by means of a vector field $v_\mu(r, x)$. We must require

$$-\tilde{\phi} = \delta\tilde{\phi} = -v_5\phi' \;\rightarrow\; v_5 = \frac{\tilde{\phi}}{\phi'} \,. \tag{5.12}$$

This transformation will introduce a nonzero h_{55} and modify h and H. We find

$$\delta h = -8A'(r)v_5 = -8\frac{A'(r)}{\phi'(r)}\tilde{\phi} = \frac{16}{3}\frac{W\tilde{\phi}}{\partial W/\partial\phi} = -\frac{8}{\beta_\phi}\,\tilde{\phi}\,. \tag{5.13}$$

For the GPPZ flow the axial-gauge solution was [43]

$$\begin{aligned}
\tilde{m}_{\text{axial}} &= \frac{\sqrt{1-u}}{u}\left[4F_1 - f_0 + p^2L^2uF_2\right],\\
h_{\text{axial}} &= \frac{1}{3\sqrt{3}u}\Big(24f_0 - 96F_1 - 24L^2p^2uF_2 + 24L^2p^2u(1-u)F_3 + \\
&\qquad + L^2p^2u^2(1-u)(8-L^2p^2)F_4\Big),\\
&= \frac{1}{3\sqrt{3}u}\left(24f_0 - 96F_1\right),
\end{aligned} \tag{5.14}$$

where we have not reproduced the solution for H, and f_0 is an integration constant associated to the pure gauge solution (5.11). In the last line we used a hypergeometric identity [61]. The hypergeometric functions $F_n\,(u;p)$ are defined by

$$F_n(u;p) \equiv F\left(n - \frac{3}{2} + \frac{1}{2}\,q, n - \frac{3}{2} - \frac{1}{2}\,q; n; u\right), \qquad q = \sqrt{1+p^2L^2}\,. \tag{5.15}$$

Passing to AFT gauge, we obtain

$$\delta h = \frac{8}{\sqrt{3}}\frac{1}{\sqrt{1-u}}\,\tilde{m} = \frac{8}{\sqrt{3}u}\left(-f_0 + 4F_1 + p^2L^2uF_2\right). \tag{5.16}$$

giving

$$h_{AFT} = \frac{8}{\sqrt{3}}p^2L^2F_2\,, \tag{5.17}$$

Up to irrelevant overall factors, this is precisely the solution for h obtained by AFT.

For the Coulomb branch flow, the solution in axial gauge was [43][9]

$$
\begin{aligned}
\tilde{\varphi}_{\rm axial} &= v^a(1-v)\ {}_3F_2\left(1+a, 2+a, \frac{1}{3}+a; 2+2a, \frac{4}{3}+a; v\right), \\
h_{\rm axial} &= \frac{4\sqrt{2}v^a\ell^2}{3\sqrt{3}L^4p^2}\Big[4(1+3a)(2-4v-v^2+a(2-v-v^2))F(1+a,2+a;2+2a;v)+ \\
&\quad + 2v(1-v)(2+v)(2+a)(1+3a)F(2+a,3+a;3+2a;v)+ \\
&\quad + 3(2+v)\frac{L^4p^2}{\ell^2}\ {}_3F_2\left(1+a,2+a,\frac{1}{3}+a;2+2a,\frac{4}{3}+a;v\right)\Big], \qquad (5.18)
\end{aligned}
$$

where ${}_mF_n(b_1,\ldots,b_m;c_1,\ldots,c_n;z)$ denotes a generalized hypergeometric function [61] and $a \equiv (-1+\sqrt{1-p^2L^4/\ell^2})/2$. Transforming into AFT gauge gives

$$
\begin{aligned}
\delta h &= -\frac{4\sqrt{6}}{3}\frac{v+2}{1-v}\tilde{\phi} = -\frac{4\sqrt{6}}{3}v^a(v+2)\ {}_3F_2\left(1+a,2+a,\frac{1}{3}+a;2+2a,\frac{4}{3}+a;v\right) \\
h_{AFT} &= -\frac{4\sqrt{2}\ell^2v^a}{3\sqrt{3}L^4p^2}(1+3a)\Big\{4[-2+4v+v^2+a(-2+v+v^2)]F(1+a,2+a;2+2a;v)- \\
&\quad - 2v(1-v)(2+v)(2+a)F(2+a,3+a;3+2a;v)\Big\}, \qquad (5.19)
\end{aligned}
$$

where the unpleasant ${}_3F_2$ hypergeometric function exactly cancels. The solution (5.19) satisfies the AFT equation of motion, [44, eq. (2.32)] , where s is related to h_{AFT} by (5.20).

5.2 Dynamical scalar gauge

Finally, we wish to discuss a third gauge, where the active scalar is kept as the dynamical degree of freedom. AFT claim that by extracting an r-dependent factor from h, one is left with a field s which is the scalar coupling to $\mathcal{O}_\phi$:

$$
h_{AFT} = -\frac{8}{\beta_\phi}\, s\,, \qquad (5.20)
$$

where we added the minus sign to accord with our definition of the β-function (5.3). One can show explicitly that this identification is correct by transforming from AFT gauge, where $\tilde{\phi}=0$, to a gauge where $h=0$. We must require

$$
-h = \delta h = -8A'(r)v_5 \;\to\; v_5 = \frac{h}{8A'(r)}\,. \qquad (5.21)
$$

Then

$$
\tilde{\phi}_{DS} = -v_5\phi'(r) = -\frac{\phi'(r)}{8A'(r)}\, h_{AFT} = \frac{3}{16W}\frac{\partial W}{\partial\phi}h_{AFT} = -\frac{\beta_\phi}{8}h_{AFT}\,. \qquad (5.22)
$$

[9] An extraneous factor of $1/(a-1/3)$ multiplying h that appeared in the first version of [43] has been corrected.

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and by virtue of (5.20),

$$\tilde{\phi}_{DS} = s\,. \tag{5.23}$$

This third gauge, in which the dynamics of the system is carried by the active scalar, is perhaps the most intuitive; we will use something similar in the fermion sector. It also seems likely that this gauge will generalize most easily to the case of several active scalars, since it involves a condition on h instead of on the set of scalars.

Using the well-established procedure for calculating 2-point functions of operators dual to scalar fields [65, 66], we find for the GPPZ case

$$\langle \mathcal{O}_{\tilde{m}}(p)\mathcal{O}_{\tilde{m}}(-p)\rangle = \frac{p^2}{2}\left[\psi\left(\frac{3}{2}+\frac{1}{2}\sqrt{1+p^2L^2}\right)+\psi\left(\frac{3}{2}-\frac{1}{2}\sqrt{1+p^2L^2}\right)\right], \tag{5.24}$$

which agrees with [44], eqn. (2.26) up to normalization. The apparent difference in the arguments of the ψ functions only corresponds to the addition of a contact term. The spectrum is $p^2 = 4(n+1)(n+2)/L^2$, which will prove to be common to all members of the anomaly multiplet, as in equation (1.2). Note that there is no massless pole.

For the Coulomb branch case, the active scalar corresponds to a $\Delta = 2$ operator, and so we must use the modified prescription for the 2-point function, see [67]. We obtain

$$\begin{aligned}\langle \mathcal{O}_{\tilde{\varphi}}(p)\,\mathcal{O}_{\tilde{\varphi}}(-p)\rangle &= -\lim_{\epsilon\to 0}\left(\epsilon^4\log^2\epsilon\right)\left[\frac{1}{z^3}\frac{d}{dz}\ln(\tilde{\varphi}(z,p))\right]_{z=\epsilon}\\ &= \psi\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{p^2L^4}{\ell^2}}\right),\end{aligned} \tag{5.25}$$

with the z-variable defined in [43]. This functional form is ubiquitous for Coulomb branch 2-point functions, and indicates the mass gap at $m^2_{\rm gap} = \ell^2/L^4$ and overlying continuum.

Returning to general considerations, one may notice that although the boundary scaling of $\tilde{\phi}$ is the same as that of any scalar field acting as a source, $\tilde{\phi} \sim e^{(\Delta-4)r}$, the behavior of the field h in AFT gauge is somewhat unusual. If the background corresponds to an operator deformation of $\mathcal{N}=4$ SYM, then the β-function scales as $\beta \sim e^{(\Delta-4)r}$, and we have

$$h_{AFT} \sim \text{const.}\,, \tag{5.26}$$

near the boundary. On the other hand, if the background corresponds to a different vacuum, $\beta \sim e^{-\Delta r}$ and we find for $\Delta > 2$,

$$h_{AFT} \sim e^{(2\Delta-4)r}\,, \tag{5.27}$$

which diverges on the boundary.[10] Thus fluctuations of h are constant in an operator background, but diverge for the vev background.

[10]For the special case of $\Delta = 2$, we have $h \sim r$ for a Coulomb background, which is also divergent.

In the Coulomb branch, the stress-tensor trace $T^{\mu}_{\mu} = 0$ vanishes, rather than being determined by the operator $\mathcal{O}_{\phi}$ as it is on an operator flow. The relation between this vanishing and the divergent behavior of (5.27) is not clear to us.

6. The vector/scalar sector

This section is devoted to studying the spectrum of SO(3)-invariant scalar fluctuations around the GPPZ flow [16] with $\sigma = 0$ in the background. As discussed in section 3, the eight real scalars, denoted m, σ and $t_{(ij)}$ in the Iwasawa/solvable parameterization, describe the quaternionic coset $G_{2(2)}/\,\mathrm{SO}(4)$.

The active scalar m and its "imaginary partner" $t_{(10)}$ sit in the active hypermultiplet. They are dual to the F-component of the anomaly multiplet $\mathcal{A} = \mathrm{Tr}(\Phi^i)^2$ in the $\mathcal{N} = 1$ superfield description of the mass-deformed field theory, namely the operator $\mathrm{Tr}(\frac{1}{2}\psi_i\psi_i + z_i\,F_i)$ of dimension $\Delta = 3$. The bulk fields m and t_{10} mix with the (trace of the) graviton and the (longitudinal component of the) graviphoton, respectively.[11] The scalar σ and its imaginary partner $t_{(32)}$ reside in the dilaton hypermultiplet, and are dual to the $\Delta = 3$ gaugino bilinear operator in the lowest component of the lagrangian multiplet $\mathcal{S} = \mathrm{Tr}(W_{\alpha}W^{\alpha}) + \cdots$

The remaining four scalar singlets mix in pairs, $t_{(11)}/t_{(31)}$ and $t_{(21)}/t_{(01)}$, but do not mix with other fields. We will exploit SUSY to diagonalize these fields, and we find that each pair contributes a real scalar dual to a $\Delta = 2$ operator in the active hypermultiplet, and a real scalar dual to a $\Delta = 4$ operator in the dilaton multiplet. The two $\Delta = 4$ fields are the 5D axion/dilaton τ, whose kinetic term is non-canonical and whose AdS mass is only asymptotically vanishing near the boundary. Note that there is no physical "massless" scalar field whose spectrum of fluctuations would coincide with that of the transverse components of the traceless graviton and graviphoton.

We will first consider the scalars that do not mix with the graviton and the graviphoton. For our purposes we need the quadratic scalar lagrangian and the linearized SUSY transformation rules. Two of the scalars, σ and $t_{(32)}$, have diagonal, canonical kinetic terms; the equation for σ has already been solved in [43] and $t_{(32)}$ is handled here analogously. In order to diagonalize the equations governing the fluctuations of $t_{(11)}/t_{(31)}$ and $t_{(21)}/t_{(01)}$, we rely on an *ansatz* suggested by the preserved supersymmetry. We then resolve the vector/pseudoscalar mixing by performing a field redefinition that brings the relevant lagrangian into Stückelberg form. The graviton/active scalar mixing has been discussed at length in section 5.

[11] As we will see momentarily, it would be more appropriate to say that the "modulus" χ of the complex active scalar mixes with the graviton and its "phase" β mixes with the graviphoton.

6.1 Free scalar equations of motion

The quadratic lagrangian for the scalar fields mixing only among themselves can be compactly written in the form

$$\frac{1}{\sqrt{g}}\mathcal{L} = \frac{1}{2}G_{IJ}(m)\partial\tilde{\phi}^I\partial\tilde{\phi}^J - \frac{1}{2}M^2_{IJ}(m)\tilde{\phi}^I\tilde{\phi}^J\,, \tag{6.1}$$

where as always $\tilde{\phi}^I$ denote the fluctuations around the classical solution. The symmetric tensors G_{IJ} and M^2_{IJ} are functions of r though their dependence on the active scalar m. Looking for solutions of the form,

$$\tilde{\phi}^I(r,x) = \tilde{\phi}^I(r,p)e^{-ip\cdot x}\,, \tag{6.2}$$

we obtain in the RG flow background the fluctuation equations

$$\partial_r^2\tilde{\phi}^I + 4A'\partial_r\tilde{\phi}^I + T^I_{rJ}\partial_r\tilde{\phi}^J + e^{-2A}p^2\tilde{\phi}^I - Z^I{}_J\tilde{\phi}^J = 0\,, \tag{6.3}$$

where $T^I_{rJ} \equiv G^{IK}\partial_r G_{JK}$ and $Z^I{}_J \equiv G^{IK}M^2_{JK}$.

As is the case with other fluctuation equations in this background, it is convenient to switch to the radial variable u (2.15). One finds

$$u(1-u)\partial_u^2\tilde{\phi}^K + \left[(2-u)\delta^K{}_J + \frac{u}{1-u}T^K_{uJ}\right]\partial_u\tilde{\phi}^J + \left[\frac{p^2L^2}{4}\delta^K{}_J - \frac{L^2}{4}\frac{u}{1-u}Z^K_J\right]\tilde{\phi}^J = 0\,. \tag{6.4}$$

We now apply equation (6.4) to each of the scalar sectors.

The fields σ and $t_{(32)}$ appear diagonally in the quadratic lagrangian.[12] They have canonical metric $G_{\sigma\sigma} = G_{(32)(32)} = 1$ and the same mass term,

$$M^2_{\sigma\sigma} = M^2_{(32)(32)} = \frac{3}{L^2}\left[1 - 2\cosh\left(\frac{2m}{\sqrt{3}}\right)\right] = \frac{3}{L^2}\left(\frac{3u-4}{u}\right)\,, \tag{6.5}$$

and thus have an identical fluctuation equation,

$$u(1-u)\sigma'' + (2-u)\sigma' + \left(\frac{p^2L^2}{4} - \frac{3(3u-4)}{4(1-u)}\right)\sigma = 0\,. \tag{6.6}$$

which was solved in [43], giving

$$\{\sigma, t_{(32)}\} = (1-u)^{3/2}\,F\left(\frac{3}{2} + \frac{1}{2}\sqrt{9+p^2L^2}, \frac{3}{2} - \frac{1}{2}\sqrt{9+p^2L^2}; 2; u\right)\,. \tag{6.7}$$

This function has poles where $3 \pm \sqrt{9+p^2L^2} = -2n$, n integer. The spectrum of poles is $p^2L^2 = 4n(n+3)$, as in equation (1.3), including a massless state.

[12] For GPPZ flows with $\sigma \neq 0$ the situation is slightly more involved. σ has to be treated at the full non-linear level since it mixes with the graviton. $t_{(32)}$ or better the phase of the *a priori* complex σ mixes with the graviphoton.

The scalars $t_{(11)}$, $t_{(31)}$, $t_{(21)}$ and $t_{(01)}$ form two 2×2 independent sectors with diagonal non-canonical metric and non-diagonal mass matrices, as given in equations (3.12), (3.15). For compactness of notation we often put $\mu = 2m/\sqrt{3}$ in the following.

In the $t_{(11)}/t_{(31)}$ sector, it is convenient to put $\tilde{\phi}^1_+ \equiv t_{(11)}$ and $\tilde{\phi}^2_+ \equiv t_{(31)}$; similarly, in the $t_{(21)}/t_{(01)}$ sector, we define $\tilde{\phi}^1_- \equiv t_{(21)}$ and $\tilde{\phi}^2_- \equiv t_{(01)}$. One then finds the functions

$$\begin{aligned}
G_{11} &= e^{\pm\mu} \qquad & M^2_{11} &= \frac{1}{4}(-10 + 11e^{\pm\mu} - 6e^{\pm 2\mu} + e^{\pm 3\mu}) \\
G_{22} &= e^{\mp 3\mu} \qquad & M^2_{22} &= \frac{1}{4}(-3e^{\mp\mu} - 12e^{\mp 2\mu} + 3e^{\mp 3\mu}) \\
G_{12} &= 0 \qquad & M^2_{12} &= \frac{\sqrt{3}}{4}(1 - 6e^{\mp\mu} + e^{\mp 2\mu}) \,.
\end{aligned} \tag{6.8}$$

It is remarkable that these two sectors are simply related by a change of sign of μ. Thus, performing the analysis in one of the two sectors, say $t_{(11)}/t_{(31)}$, suffices for both. The equations of motion are then of the form

$$\tilde{\phi}^{I\prime\prime} + (4A' + a_I)\tilde{\phi}^{I\prime} + e^{-2A}p^2\tilde{\phi}^I - b_I\tilde{\phi}^I - c_I\tilde{\phi}^{\bar{I}} = 0 \,, \tag{6.9}$$

where $a_I \equiv d\log(G_{II})/dr$, $b_I \equiv (G_{II})^{-1}M_{II}$ and $c_I \equiv (G_{II})^{-1}M_{I\bar{I}}$ with $\bar{I} = I + 1$ (mod 2). One can easily get an uncoupled fourth-order equation for one of the scalars by eliminating its partner. However, exploiting the linearized SUSY preserved by the kink solution, we can deduce an ansatz for the scalar fluctuations that turns out to reduce the problem to standard second-order ordinary differential equation.

Consider the SUSY transformations involving $t_{(11)}$ and $t_{(31)}$ in (3.27), (3.28). Their form strongly suggests that it is the separate linear combinations, whose transforms involve $\xi^{1,2}$ and $\xi^{3,4}$ respectively, which satisfy uncoupled scalar equations. A similar observation can be made for $t_{(21)}$ and $t_{(01)}$. This leads us to define

$$\begin{aligned}
\tilde{\phi}^1_\pm &\equiv e^{\mp\mu/2}(\rho_\pm - \sqrt{3}\tau_\pm) \,, \\
\tilde{\phi}^2_\pm &\equiv e^{\pm 3\mu/2}(\sqrt{3}\rho_\pm + \tau_\pm) \,.
\end{aligned} \tag{6.10}$$

Indeed we will find that ρ and τ decouple and satisfy second order equations whose mass terms $(mL)^2$ approach -4 and 0, respectively, as $r \to \infty$. Thus ρ, τ are dual to operators of dimension $\Delta = 2, 4$ respectively.

To verify decoupling we substitute the combinations (6.10) into the fluctuation equations (6.9) and *assume* that the resulting equations are satisfied independently by ρ and τ. One then finds two second order equations for each of the four modes $\rho_\pm, \tau_\pm$ with complicated and apparently different effective mass terms. Miraculously, when the background flow equations (2.7), (2.10) and the GPPZ superpotential (2.13) are used, the two equations are seen to be equivalent, which *proves* the consistency of the assumed decoupling. A further simplification is that the equations for the

$\pm$ modes are even in μ and thus identical! For $\rho = \rho_+ = \rho_-$ the final fluctuation equation takes the relatively simple form,

$$\rho'' + 4A'\rho' + \left(e^{-2A}p^2 + \frac{1}{4L^2}\left[9 + 10\cosh(\mu) - 3\cosh(\mu)^2\right]\right)\rho = 0\,. \qquad (6.11)$$

In the variable u this becomes

$$u(1-u)\rho'' + (2-u)\rho' + \left(\frac{p^2L^2}{4} - \frac{u^2 - 8u + 3}{4u(1-u)}\right)\rho = 0\,, \qquad (6.12)$$

which has the solution

$$\rho = u^{1/2}(1-u)\,F\left(\frac{3}{2} + \frac{1}{2}\sqrt{1+p^2L^2}, \frac{3}{2} - \frac{1}{2}\sqrt{1+p^2L^2}; 3; u\right), \qquad (6.13)$$

The spectrum of poles is at $3 \pm \sqrt{1+p^2L^2} = -2n$, giving $(pL)^2 = 4(n+1)(n+2)$, as appropriate for the multiplet $\mathcal{A}$, see equation (1.2).

The active scalar $\tilde{m}$, reviewed in section 5, and its partner $t_{(10)}$, which we examine in the next subsection, share this spectrum of poles with ρ_+ and ρ_-, leading to a fourfold degeneracy. This is exactly what one expects on SUSY grounds, as m and $t_{(10)}$ are dual to the top component $F_{\mathcal{A}}$ of the anomaly multiplet $\mathcal{A}$, with $\Delta = 3$, while $\rho_\pm$ are dual to the lowest component $\phi_{\mathcal{A}}$ with $\Delta = 2$. The mixing of $\tilde{m}$ and $t_{(10)}$ with h^μ_μ and $\partial^\mu B_\mu$ is dual to the fact that $F_{\mathcal{A}}$ contains T^μ_μ and $\partial^\mu R_\mu$, while the fact that $\rho_\pm$ do not mix with the gravity multiplet is appropriate since $\phi_{\mathcal{A}}$ does not correspond to any mode of the stress tensor or R-current. The fermionic components of this multiplet, ξ^1 and ξ^2, will be shown to have the same spectrum in section 7.

For the fields $\tau = \tau_+ = \tau_-$ one obtains equations equivalent to

$$\tau'' + 4A'\tau' + \left(e^{-2A}p^2 - \frac{1}{4L^2}\left[15 - 18\cosh(\mu) + 3\cosh(\mu)^2\right]\right)\tau = 0\,, \qquad (6.14)$$

which, thanks to the symmetry between the two ($+$ and $-$) sectors, is even in μ. In the u-variable one gets

$$u(1-u)\tau'' + (2-u)\tau' + \left(\frac{p^2L^2}{4} + \frac{3(3u-1)}{4u}\right)\tau = 0\,, \qquad (6.15)$$

which is solved by

$$\tau = u^{1/2}(1-u)^2 F\left(\frac{5}{2} + \frac{1}{2}\sqrt{9+p^2L^2}, \frac{5}{2} - \frac{1}{2}\sqrt{9+p^2L^2}; 3; u\right). \qquad (6.16)$$

The spectrum of poles is $(pL)^2 = 4(n+1)(n+4)$, identical to that of the scalar fields $t_{(32)}$ and σ except for the absence of the zero-mass pole. Indeed the four scalars in question are dual to the two complex scalar components of the chiral multiplet $\mathcal{S} = \mathrm{Tr}(W^2) + \cdots$. The scalar fields σ and $t_{(32)}$ are dual to the lowest component $\phi_{\mathcal{S}}$ with $\Delta = 3$, while the fluctuations $\tau_\pm$ have asymptotic mass $m^2L^2 = 0$ and are dual to the top component $F_{\mathcal{S}}$ with $\Delta = 4$.

The fact that the massive poles are fourfold degenerate while the massless pole is only doubly degenerate is explained by the following supersymmetry argument. The two-point function of a gauge-invariant chiral superfield in coordinate superspace $(x, \theta, \bar{\theta})$ is fixed by $\mathcal{N}=1$ supersymmetry to be of the form

$$\langle \mathcal{S}(x_1, \theta_1, \bar{\theta}_1) \mathcal{S}^\dagger(x_2, \theta_2, \bar{\theta}_2) \rangle = e^{i(\theta_1 \sigma \bar{\theta}_1 + \theta_2 \sigma \bar{\theta}_2 - 2\theta_1 \sigma \bar{\theta}_2)\cdot \partial_1} \Delta(x_{12}) \,, \tag{6.17}$$

where $\Delta(x_{12})$ is an *a priori* arbitrary scalar function of the relative position $x_{12}^\mu = x_1^\mu - x_2^\mu$. After Fourier transforming and expanding in powers of θ one easily obtains the "SUSY Ward identity"

$$\langle F_{\mathcal{S}}(p) F_{\mathcal{S}}^\dagger(-p) \rangle = p^2 \langle \phi_{\mathcal{S}}(p) \phi_{\mathcal{S}}^\dagger(-p) \rangle \,. \tag{6.18}$$

We thus see that any potential simple pole at $p^2 = 0$ for the lowest component $\phi_{\mathcal{S}}$ is cancelled by the factor of p^2 for the top component $F_{\mathcal{S}}$.[13] As we will see in section 7, the multiplet is completed by the addition of the fermionic partners ξ^3, ξ^4, which decouple from the gravitino and present the expected spectrum, including the massless pole for one of the two chiralities.

6.2 Resolving the vector/scalar mixing

Let us now discuss the mixing of the Goldstone field $t_{(10)}$ with the graviphoton field B_μ. The key observation here is that the four scalar fields $\{m, t_{(10)}, \sigma, t_{(32)}\}$ parameterize an $\mathrm{SL}(2)/\mathrm{U}(1) \times \mathrm{SL}(2)/\mathrm{U}(1)$ submanifold of $G_{2(+2)}/\mathrm{SO}(4)$ [53, 54] and the $\mathrm{U}(1)_R$ generator defined in (3.17) is a linear combination of the compact $\mathrm{U}(1)$ isometries of the two $\mathrm{SL}(2)/\mathrm{U}(1)$ factors. Since we are studying a flow with $\sigma(r) = 0$, there are no bilinear mixing terms from the second $\mathrm{SL}(2)/\mathrm{U}(1)$ factor, and we can restrict attention to the first factor only. We can derive all the information we need from a standard lagrangian for the gauged $\mathrm{SL}(2)/\mathrm{U}(1)$ σ-model,

$$\frac{1}{\sqrt{g}} \mathcal{L}_{\text{gauged}} = -\frac{3}{4} F^2 + \frac{3}{8} \left[\partial \chi^2 + \sinh^2(\chi) (\partial \beta - g B)^2 \right] , \tag{6.19}$$

in which β is the angular variable of the compact $\mathrm{U}(1)$ isometry.

We first demonstrate the equivalence of (6.19) to the form given in section 3. The change of coordinates

$$\cosh(\chi) = \frac{X^2 + Y^2 + 1}{2Y} \,, \qquad \tan(\beta) = \frac{2X}{1 - X^2 - Y^2} \,, \tag{6.20}$$

takes us to the Poincaré plane form

$$\frac{1}{\sqrt{g}} \mathcal{L}_{\text{gauged}} = -\frac{3}{4} F^2 + \frac{3}{8Y^2} \left[(\partial Y - g B X Y)^2 + [\partial X + \frac{g}{2} B (1 - Y^2 - X^2)]^2 \right] \,. \tag{6.21}$$

[13] Notice that $F_{\mathcal{S}}$ is auxiliary/non-propagating, i.e. has δ-function propagator in coordinate space, only for a free chiral superfield.

The further transformation

$$Y = e^{\frac{2m}{\sqrt{3}}} \qquad X = \frac{2}{\sqrt{3}} t_{(10)} \tag{6.22}$$

gives a kinetic lagrangian which agrees to quadratic order with (3.12), (3.18), (3.19) of section 3 (when $\sigma = 0$), but is a fully nonlinear extension thereof.

The U(1) is a subgroup of the gauged SO(6) invariance of the $\mathcal{N} = 8$ supergravity theory, so the scalar potential is U(1) invariant and thus independent of the field β. We can therefore use the χ, β form of the lagrangian (6.19) for our present purpose of determining the vector/scalar fluctuations, and we can replace χ by its background value $\chi = \mu \equiv 2m/\sqrt{3}$. This gives

$$\sinh^2(\chi) = \frac{1}{2}\left[\cosh\left(\frac{4m(r)}{\sqrt{3}}\right) - 1\right] = -\frac{8A''(r)}{g^2}\,. \tag{6.23}$$

The lagrangian (6.19) to quadratic order in fluctuations of B_μ and β then becomes

$$\frac{1}{\sqrt{g}}\mathcal{L} = -\frac{3}{4}F^2 + \frac{3}{2g^2}\, m_B^2 \left(\partial\beta - gB\right)^2 , \tag{6.24}$$

where $m_B^2 = -2A''(r)$ is the graviphoton mass obtained in section 3 and we have kept the unconventional normalization factor of 3 that emerged from the truncation of $\mathcal{N} = 8$ supergravity in that section. We have thus reduced the system to a standard Stückelberg lagrangian, a fact which will help immensely as we now turn to the solution of its equations of motion,

$$D_\mu F^{\mu\nu} - \frac{m_B^2}{g}(\partial^\nu\beta - gB^\nu) = 0\,, \tag{6.25}$$

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}m_B^2(\partial_\nu\beta - gB_\nu)) = 0\,. \tag{6.26}$$

The contracted Bianchi identity for (6.25), i.e. current conservation,

$$D_\mu D_\nu F^{\mu\nu} = 0 = D_\mu J^\mu\,, \tag{6.27}$$

implies the scalar equation (6.26).

The equations are gauge invariant and we discuss their solution in two gauges. The fastest route to a solution employs the gauge $\beta(x,r) = 0$ which is the analogue of the AFT gauge of section 5. This is equivalent to the Stückelberg approach, where one works in terms of a gauge invariant vector field,

$$\mathcal{B}_\mu = B_\mu - \frac{1}{g}\partial_\mu\beta\,. \tag{6.28}$$

The transverse components, which were treated in section 4, decouple from the remaining longitudinal and radial components, and the latter are related by the current conservation condition

$$D_\mu(m_B^2\mathcal{B}^\mu) = 0 \tag{6.29}$$

or, more explicitly,

$$\eta^{ij}\partial_i\mathcal{B}_j = \frac{1}{e^{2A}m_B^2}\partial_r(e^{4A}m_B^2\mathcal{B}_r)\,. \tag{6.30}$$

We use this to write radial component of the vector equation of motion as

$$\Box\mathcal{B}_r - \partial_r\left(\frac{1}{e^{2A}m_B^2}\partial_r(e^{4A}m_B^2\mathcal{B}_r)\right) + e^{2A}m_B^2\mathcal{B}_r = 0\,. \tag{6.31}$$

One may then show that the redefined field $C(u,p) \equiv e^{4A(r)}m_B^2\mathcal{B}_r(r,p)$ satisfies

$$u(1-u)C'' + (1-2u)C' + \left(\frac{p^2L^2}{4} - \frac{1}{u}\right)C = 0\,, \tag{6.32}$$

with the hypergeometric solution, $C(u,p) = uF_3(u,p)$ in the notation of (5.14). This has the same spectrum of poles as its partner, h_{AFT} or s, in the anomaly multiplet.

We now briefly discuss the axial gauge $B_r(r,x) = 0$ which has residual gauge transformations generated by a gauge parameter $\alpha(x)$ which is independent of r but otherwise arbitrary. In this gauge the radial component of the vector equations of motion implies

$$\partial_r(\partial_i B^i) = \frac{1}{g}\,e^{2A}\,m_B^2\,\partial_r\beta\,. \tag{6.33}$$

In analogy with section 5, one can use this to obtain the uncoupled third order scalar equation

$$\partial_r\left(\frac{1}{m_B^2e^{2A}}\partial_r(e^{4A}m_B^2\partial_r\beta)\right) = -p^2\partial_r\beta + m_B^2e^{2A}\partial_r\beta\,. \tag{6.34}$$

This can be viewed as a second order equation for $\partial_r\beta$, and one can use (6.33) and (6.30) to express its solution in terms of $C(u,p)$.

Including the second SL(2)/ U(1) factor is an easy task. To the lagrangian (6.21) one simply has to add

$$\frac{1}{\sqrt{g}}\Delta\mathcal{L}_{\text{gauged}} = \frac{1}{8U^2}\left[(\partial U + 3gBUV)^2 + \left[\partial V - \frac{3g}{2}B(1-U^2-V^2)\right]^2\right], \tag{6.35}$$

where

$$U = e^{-2\sigma}\,, \quad V = 2t_{(32)}\,. \tag{6.36}$$

As remarked in section 3, $m_B^2 = -2A''$ continues to hold even when $\sigma \neq 0$. The different normalization of the $\mathrm{U}(1)_R$ charge in the $\{\sigma, t_{(32)}\}$ sector is as required by the generator in (3.17).

7. The fermion sector

We now consider the fermion sector in the GPPZ flow. In analogy to the coupling of the graviton trace and longitudinal graviphoton with the active scalar's modulus and phase, respectively, we will find that the γ-trace of the gravitino couples to the fermi fields from the active hypermultiplet. The spinors from the dilaton hypermultiplet are uncoupled and display the expected spectrum.

Setting $\sigma = 0$, the fermion lagrangian is

$$\begin{aligned} e^{-1}\mathcal{L} = & -\frac{i}{2}\left(\bar{\psi}^1_\mu\gamma^{\mu\nu\rho}D_\nu\psi^2_\rho - \bar{\psi}^2_\mu\gamma^{\mu\nu\rho}D_\nu\psi^1_\rho\right) - \frac{ig}{4}W\left(\bar{\psi}^1_\mu\gamma^{\mu\nu}\psi^1_\nu + \bar{\psi}^2_\mu\gamma^{\mu\nu}\psi^2_\nu\right) - \\ & -\frac{i}{2}\left(\bar{\xi}^1\gamma^\mu D_\mu\xi^2 - \bar{\xi}^2\gamma^\mu D_\mu\xi^1 + \bar{\xi}^3\gamma^\mu D_\mu\xi^4 - \bar{\xi}^4\gamma^\mu D_\mu\xi^3\right) - \\ & -\frac{i}{2}M(r)\left(\bar{\xi}^1\xi^1 + \bar{\xi}^2\xi^2 - 3\bar{\xi}^3\xi^3 - 3\bar{\xi}^4\xi^4\right) + \frac{m'(r)}{\sqrt{2}}\left(\bar{\psi}^1_\mu\gamma^r\gamma^\mu\xi^1 + \bar{\psi}^2_\mu\gamma^r\gamma^\mu\xi^2\right) - \\ & -\frac{\sqrt{3}g}{4\sqrt{2}}\sinh\left(\frac{2m}{\sqrt{3}}\right)\left(\bar{\xi}^2\gamma^\mu\psi^1_\mu - \bar{\xi}^1\gamma^\mu\psi^2_\mu\right), \end{aligned} \tag{7.1}$$

where

$$M(r) \equiv -\frac{3}{2}A' - \frac{m''}{m'} = \frac{g}{8}\left[\cosh\left(\frac{2m}{\sqrt{3}}\right) - 3\right]. \tag{7.2}$$

The $\delta\chi^{abc} = 0$ condition for unbroken supersymmetry also requires

$$m'(r) = -\frac{\sqrt{3}g}{4}\sinh\left(\frac{2m}{\sqrt{3}}\right), \tag{7.3}$$

which leads to the appearance of chirality projectors in the equations of motion. Defining the complex spinors

$$\psi_\mu \equiv \psi^1_\mu + i\psi^2_\mu\,, \qquad \xi \equiv \xi^1 + i\xi^2\,, \qquad \eta \equiv \xi^3 + i\xi^4\,, \tag{7.4}$$

one obtains the field equations

$$i\gamma^{\mu\nu\rho}D_\nu\psi_\rho = \frac{g}{2}W\gamma^{\mu\nu}\psi_\nu + \frac{m'}{\sqrt{2}}\left(1 + i\gamma^r\right)\gamma^\mu\xi\,, \tag{7.5}$$

$$i\gamma^\mu D_\mu\xi = M(r)\xi - \frac{m'}{\sqrt{2}}\gamma^\mu\left(1 - i\gamma^r\right)\psi_\mu\,, \tag{7.6}$$

$$i\gamma^\mu D_\mu\eta = -3M(r)\eta\,, \tag{7.7}$$

where here $D_i = \partial_i - \frac{1}{2}A'\gamma_i\gamma_r$ and $D_5 = \partial_5$; Christoffel connections cancel in (7.5) due to antisymmetry.

7.1 The uncoupled spinor

We begin by considering equation (7.7) for the uncoupled spinor field η. These techniques will generalize to the coupled ψ_μ/ξ system. The η field sits in the inert hypermultiplet with the scalars σ and the "dilaton" τ, and so should have the same spectrum.

Define the "chirality" projectors

$$P_\pm \equiv \frac{1}{2}\left(1 \pm i\gamma^r\right), \tag{7.8}$$

which obey

$$P_\pm\gamma_r = \gamma_r P_\pm\,, \qquad P_\pm\gamma_i = \gamma_i P_\mp\,, \qquad P_\pm D_\mu = D_\mu P_\pm\,. \tag{7.9}$$

The chiral projections of equation (7.7) are then

$$i\,\partial\!\!\!/\eta_+ - 2A'\eta_- + i\gamma^r\partial_r\eta_- + 3M(r)\eta_- = 0\,, \tag{7.10}$$

$$i\,\partial\!\!\!/\eta_- + 2A'\eta_+ + i\gamma^r\partial_r\eta_+ + 3M(r)\eta_+ = 0\,, \tag{7.11}$$

where $\partial\!\!\!/ \equiv \gamma^i\partial_i$. Our strategy is to eliminate one of the projections of η and solve for the other. Writing $\eta(x,r) = e^{ipx}\eta(p,r)$, we can solve for η_- using (7.11),

$$\eta_- = \frac{1}{p\!\!\!/}\left(\partial_r + 2A' + 3M\right)\eta_+\,. \tag{7.12}$$

We then substitute (7.12) into (7.10). The identity $\partial_r(1/p\!\!\!/) = A'/p\!\!\!/$ is needed. Finally we multiply by an overall $p\!\!\!/$ and use $p\!\!\!/p\!\!\!/ = g^{ij}p_ip_j$ to obtain

$$\left(-e^{-2A}p^2 - \partial_r^2 - 5A'\partial_r - 6A'^2 - 2A'' + 9M^2 - 3MA' - 3M'\right)\eta_+ = 0\,, \tag{7.13}$$

where as usual $p^2 = \eta^{ij}p_ip_j$. Thus we find the same ordinary differential equation for each spinor component.

We solve (7.13) in the u-variable defined in (2.15). One may use the flow equations (2.7), (2.10) to show that $MA' + M' = -\frac{1}{2}A'^2$. One then finds

$$\begin{aligned}\eta_+''(u) + \frac{1}{1-u}\left(\frac{5u}{2} - 1\right)\eta_+'(u) + \frac{p^2L^2}{4}\frac{1}{u(1-u)}\eta_+(u) - \\ \frac{1}{u^2(1-u)}\left(1 - \frac{9}{8}\frac{1}{1-u} + \frac{9}{16}\frac{(1-2u)^2}{1-u}\right)\eta_+(u) = 0\,.\end{aligned} \tag{7.14}$$

This equation has the solution

$$\eta_+ = u^{1/4}(1-u)^{9/4}\;F\left(\frac{5}{2} + \frac{1}{2}\sqrt{9+p^2L^2}, \frac{5}{2} - \frac{1}{2}\sqrt{9+p^2L^2}; 3; u\right)\eta_+^{(0)}(p)\,, \tag{7.15}$$

where $\eta_+^{(0)}(p)$ is an r-independent spinor. One may read off the spectrum: poles occur when $5 - \sqrt{9+p^2L^2} = -2n$, n integer, i.e. when $p^2L^2 = 4(n+1)(n+4)$. This is indeed the same spectrum as the σ scalar, as calculated in [43] and reviewed in section 6, except for the absence of the massless pole.

The η_- projection is then determined by (7.12), or alternately by deriving an equation analogous to (7.13) for η_-, by reversing the roles of η_+ and η_- throughout. We find the solution

$$\eta_- = u^{-1/4}(1-u)^{7/4}\;F\left(\frac{3}{2} + \frac{1}{2}\sqrt{9+p^2L^2}, \frac{3}{2} - \frac{1}{2}\sqrt{9+p^2L^2}; 2; u\right)\eta_-^{(0)}(p)\,, \tag{7.16}$$

where only one of $\eta_{\pm}^{(0)}(p)$ can be specified independently. The spectrum here is identical except the massless pole is present in this chirality: $p^2L^2 = 4n(n+3)$.

The leading behavior on the boundary ($u \to 1$) is

$$\eta_+ \sim (1-u)^{1/4}\,, \qquad \eta_- \sim (1-u)^{3/4}\,. \tag{7.17}$$

Thus η_+ dominates on the boundary. It displays the correct scaling for a field dual to the $\Delta = 7/2$ spinor operator in the anomaly multiplet $\mathcal{A}$, consistent with its limiting mass on the boundary, $-3M(r) \to -3/2L$.

7.2 Coupled gravitino system

We now discuss a solution to the coupled system of the gravitino ψ_μ and the spin-1/2 field ξ, equations (7.5) and (7.6). Components of the γ-trace of ψ_μ mix explicitly with ξ. This is analogous to the mixing of the graviton trace and the active scalar modulus, and that of the longitudinal graviphoton with the active scalar phase.

The first issue is gauge fixing. Since the chiral projectors appear in equations (7.5) and (7.6), it is useful to separate the supersymmetry variations into the separate chiralities:

$$\begin{aligned}
\delta\psi_{i-} &= D_i\epsilon_- + \frac{i}{2}A'\gamma_i\epsilon_+\,, & \delta\psi_{i+} &= D_i\epsilon_+ + \frac{i}{2}A'\gamma_i\epsilon_-\,,\\
\delta\psi_{5-} &= \left(D_5 + \frac{i}{2}A'\gamma_r\right)\epsilon_-\,, & \delta\psi_{5+} &= \left(D_5 + \frac{i}{2}A'\gamma_r\right)\epsilon_+\,,\\
\delta\xi_- &= i\sqrt{2}\,m'\epsilon_-\,, & \delta\xi_+ &= 0\,.
\end{aligned} \tag{7.18}$$

We observe that the gauge choice $\xi = 0$, the analogue of the AFT gauge of section 5, is not possible here since the projection ξ_+ is gauge invariant. Nor can we decouple the gravitino from (7.6), since both $\delta\gamma^i\psi_{i-}$ and $\delta\psi_{5-}$ depend only on ϵ_-. Instead, it proves useful to use the gauge freedom to eliminate the 4D trace of the gravitino,

$$\gamma^i\psi_i = 0\,. \tag{7.19}$$

which is akin to the dynamical scalar gauge $h = 0$ in the graviton sector. Any residual gauge transformation in this gauge must satisfy

$$\delta(\gamma^i\psi_i) = \left(\not{D}_4 + 4\frac{i}{2}A'\right)\epsilon = 0\,, \tag{7.20}$$

which is equivalent to

$$\not{\partial}\epsilon_- = 0\,, \qquad \not{\partial}\epsilon_+ = -4iA'\epsilon_-\,. \tag{7.21}$$

Thus there are no residual gauge transformations with arbitrary dependence on x^i.

Implementing the gauge condition (7.19), we proceed as in the analysis of η to decompose (7.6) into chiralities

$$i\,\not{\partial}\xi_- + 2A'\xi_+ + i\gamma^r\partial_r\xi_+ - M\xi_+ = 0\,, \tag{7.22}$$

$$i\,\not{\partial}\xi_+ - 2A'\xi_- + i\gamma^r\partial_r\xi_- - M\xi_- = -\sqrt{2}im'\psi_{5-}\,, \tag{7.23}$$

and solve for ξ_- using equation (7.22) to obtain

$$\xi_- = \frac{1}{\not p}\left(\partial_r + 2A' - M\right)\xi_+ \,. \tag{7.24}$$

Substituting into equation (7.23), we find

$$\left(-e^{-2A}p^2 - \partial_r^2 - 5A'\partial_r - 6A'^2 - 2A'' + M^2 + MA' + M'\right)\xi_+ = -\sqrt{2}im'\not p\psi_{5-} \,. \tag{7.25}$$

Thus we can solve for ξ_+ (and implicitly ξ_-) if we can eliminate ψ_{5-} from this expression. The gravitino equation (7.5) allows us to do this. In performing the analysis, we must take care not to impose the gauge condition (7.19) until after all covariant derivatives have acted, so as not to throw out nonzero terms coming from connections. We then find

$$\begin{aligned}\sqrt{2}m'P_+\gamma^\mu\xi = {} & iA'\delta_j^\mu(\gamma^j\psi_5 + \gamma_r\psi^j) - i\gamma^\mu\partial^j\psi_j + i\gamma^\mu \not D_5\gamma^r\psi_5 - \\ & - iD^\mu\gamma^r\psi_5 + \frac{3}{2}A'\gamma^\mu\gamma^r\psi_5 + i\gamma^\mu D_5\psi_5 + i\not D_5\psi^\mu - \frac{3}{2}A'\psi^\mu \,.\end{aligned} \tag{7.26}$$

Setting $\mu = 5$, we find after a number of cancellations,

$$-i\gamma^r\partial^j\psi_j = \sqrt{2}m'P_+\gamma^r\xi \,, \tag{7.27}$$

leading to the chiral equations

$$\partial^j\psi_{j-} = 0 \,, \qquad \partial^j\psi_{j+} = \sqrt{2}im'\xi_+ \,. \tag{7.28}$$

Furthermore, computing the γ-trace of the $\mu = i$ component of (7.26) leads to:

$$-2i\partial^j\psi_j + 3i\not D_4\gamma^r\psi_5 + 6A'\gamma^r\psi_5 = 4\sqrt{2}m'\xi_- \,. \tag{7.29}$$

Taking the P_+ projection and using equations (7.28), we find

$$\not\partial\psi_{5-} = \frac{2\sqrt{2}}{3}m'\xi_+ \,. \tag{7.30}$$

The P_- projection determines ψ_{5+} entirely in terms of ξ, as well.

Using the relation (7.30), we may now eliminate ψ_{5-} from (7.25) and reach our goal, an uncoupled ordinary differential equation for ξ_+. We find

$$\left(-e^{-2A}p^2 - \partial_r^2 - 5A'\partial_r - 6A'^2 - 2A'' + M^2 + MA' + M' + \frac{4}{3}m'^2\right)\xi_+ = 0 \,. \tag{7.31}$$

In the u-coordinate, this becomes

$$\begin{aligned}\xi_+''(u) + \frac{1}{1-u}\left(\frac{5u}{2} - 1\right)\xi_+'(u) + \frac{p^2L^2}{4}\frac{1}{u(1-u)}\xi_+(u) - \\ - \frac{1}{u^2(1-u)}\left(2 - \frac{13}{8}\frac{1}{1-u} + \frac{1}{16}\frac{(1-2u)^2}{1-u}\right)\xi_+(u) = 0 \,,\end{aligned} \tag{7.32}$$

which has the solution

$$\xi_+(u) = u^{1/4}(1-u)^{5/4}\ F\left(\frac{3}{2}+\frac{1}{2}\sqrt{1+p^2L^2}, \frac{3}{2}-\frac{1}{2}\sqrt{1+p^2L^2}; 3; u\right)\xi_+^{(0)}(p)\,. \quad (7.33)$$

The spectrum here is $p^2L^2 = 4(n+1)(n+2)$, which agrees with other components of the anomaly multiplet.

We may compute ξ_- from equation (7.24). We obtain

$$\xi_-(u) = u^{-1/4}(1-u)^{7/4}\Bigg[12F\left(\frac{3}{2}+\frac{1}{2}\sqrt{1+p^2L^2}, \frac{3}{2}-\frac{1}{2}\sqrt{1+p^2L^2}; 3; u\right)+ \quad (7.34)$$
$$+u(8-p^2L^2)F\left(\frac{5}{2}+\frac{1}{2}\sqrt{1+p^2L^2}, \frac{5}{2}-\frac{1}{2}\sqrt{1+p^2L^2}; 4; u\right)\Bigg]\,\xi_-^{(0)}(p)\,.$$

The fields ψ_5 and $\partial^j\psi_j$ are then obtained algebraically from the ξ solutions. Combined with the gauge choice (7.19), the only components of ψ_μ not determined by ξ are the transverse, γ-traceless modes that we analyzed in section 4.

The behavior of the solutions near the boundary is

$$\xi_- \sim (1-u)^{3/4}\,, \qquad \xi_+ \sim (1-u)^{5/4}\log(1-u)\,. \quad (7.35)$$

In this case the negative-chirality component dominates. It has the correct behavior for a field dual to a $\Delta = 5/2$ operator, as implied by its limiting mass $M(r) \to 1/2L$.

7.3 Bianchi Identity

As in the graviton and gauge field systems, the presence of a local symmetry implies a Bianchi-like identity. The supersymmetry variation of the fermionic action must vanish:

$$\delta_\epsilon S = \int d^5x\left(\delta\bar\psi_\mu\frac{\partial\mathcal{L}}{\partial\bar\psi_\mu} + \delta\bar\xi\frac{\partial\mathcal{L}}{\partial\bar\xi}\right) = 0\,. \quad (7.36)$$

However, $\partial\mathcal{L}/\partial\bar\psi_\mu = \partial\mathcal{L}/\partial\bar\xi = 0$ are the equations of motion; a certain linear combination of these equations is thus trivial owing to the gauge symmetry of the system.

Integrating by parts, we find that

$$\delta_\epsilon S = -\int d^5x\bar\epsilon\left((D_\mu + \frac{i}{2}A'\gamma_\mu)\frac{\partial\mathcal{L}}{\partial\bar\psi_\mu} + \sqrt{2}im'P_+\frac{\partial\mathcal{L}}{\partial\bar\xi}\right)\,. \quad (7.37)$$

Let us act on the gravitino equation (7.5) with the operator $D_\mu + \frac{i}{2}A'\gamma_\mu$, without specializing to any particular gauge. Note that one must be careful acting on the r.h.s. of (7.5) with D_μ. There are two terms that combine to form a projector, $\frac{m'}{\sqrt{2}}\gamma^\mu\xi$ and $i\frac{m'}{\sqrt{2}}\gamma^r\gamma^\mu\xi$. The first term comes from the Lorentz scalar A_{abcd}, but the second comes from $P_{5\,abcd}\gamma^5$. Before acting on the second term with the covariant derivative, we must restore it to covariant form

$$i\frac{m'}{\sqrt{2}}\gamma^r\gamma^\mu\xi \to i\frac{(\partial_\nu m)}{\sqrt{2}}\gamma^\nu\gamma^\mu\xi\,. \quad (7.38)$$

The covariant derivative acting on $\partial_i m$ creates additional, nonvanishing terms proportional to m'.

One then obtains

$$\left(D_\mu + \frac{i}{2}A'\gamma_\mu\right)\frac{\partial\mathcal{L}}{\partial\bar\psi_\mu} = -i\sqrt{2}m'P_+\left[i\gamma^\mu D_\mu\xi + \left(\frac{3}{2}A' + \frac{m''}{m'}\right)\xi + \frac{m'}{\sqrt{2}}(1+i\gamma^r)\gamma^i\psi_i\right]. \tag{7.39}$$

Using equation (7.2), we see this is precisely $-\sqrt{2}im'P_+\frac{\partial\mathcal{L}}{\partial\bar\xi}$. Thus the Bianchi identity is satisfied; we may consider one chirality of the ξ equation as a consequence of the gravitino equation of motion.

8. Coulomb branch fluctuations

We turn now to an examination of the $n = 2$ Coulomb branch flow, as reviewed in section 2.2, an example of the class of flow backgrounds that modify the field theory vacuum. As discussed in section 5, the coupling of the graviton trace and active scalar is universal for all RG flow backgrounds. Here we will examine the coupled gravitino/spin-1/2 sectors, as well as gauge fields corresponding to unbroken R-symmetries. We will find that the fermions also have the same equations of motion as in the GPPZ case, which is presumably a consequence of $\mathcal{N} = 1$ SUSY. The graviphoton, however, behaves differently: it remains massless, as the dual R-current is unbroken. Instead, its kinetic terms are modified by the active scalar. We will demonstrate through field redefinition that the modified kinetic terms are exactly equivalent to the mass that arose in the GPPZ case.

The differences in the graviphoton sector can be traced to the fact that the active scalar φ is real and sits in an $\mathcal{N} = 2$ vector multiplet. Unlike the operator flow case, there is no phase to play the role of a Goldstone boson — indeed, there is no other scalar at all in the multiplet — and consequently the graviphoton is left unhiggsed. Though the R-current is preserved, conformal symmetry is (spontaneously) broken; it is known [47] that in such a case the field theory anomaly multiplet is a linear multiplet rather than a chiral multiplet. Hence in this case the bulk (active) vector multiplet is dual to the operators of a linear multiplet.

The 2-point function for the TT graviton (identical to the dilaton in this case) was first considered in [10], where it was found to have a continuous spectrum with a mass gap $m^2_{\rm gap} = \ell^2/L^4$. The inert scalar σ [43], as well as the $h/\tilde\varphi$ system [43, 44] which was finally solved in section 5.2, share the continuum and gap. All the fluctuations we will solve for in this section have the same features.

8.1 Truncation to $\mathcal{N} = 4$ supergravity

The equations of motion can be obtained as usual from the parent $\mathcal{N} = 8$ theory — an easier process than for the GPPZ flow, since the active scalar sits in the simpler $\mathrm{SL}(6,\mathbb{R})/\,\mathrm{SO}(6)$ submanifold of the $E_{6(6)}/\,\mathrm{USp}(8)$ scalar coset. However, because the large unbroken supersymmetry fully determines the equations of the fields we are interested in, it is even easier to obtain them through a truncation to the 5D $\mathcal{N}=4$ gauged supergravity theory considered by Romans [68].

In the $\mathcal{N}=4$ theory, the gravity multiplet is the sum of $\mathcal{N}=2$ gravity, gravitino and vector multiplets, and contains a graviton, two pairs of symplectic Majorana gravitini, a single real scalar, and gauge fields for the $\mathrm{SU}(2)\times\mathrm{U}(1)$ R-symmetry, as well as 2-forms and two pairs of spin-1/2 fields. The preserved R-symmetry of the background is $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\times\mathrm{U}(1)_R\subset\mathrm{SU}(4)$, and we are free to choose an $\mathcal{N}=4$ subalgebra of the full $\mathcal{N}=8$ that has $\mathrm{SU}(2)_R\times\mathrm{U}(1)_R$ as its R-symmetry. The real scalar in the $\mathcal{N}=4$ multiplet must be an R-singlet; however, in this embedding the active scalar is the unique real scalar with these quantum numbers. Hence the active scalar sits in our $\mathcal{N}=4$ gravity multiplet, and we can use the lagrangian of [68] to read off the couplings of any fields that also fall in this multiplet.

In this $\mathcal{N}=4$ subalgebra, the $\mathrm{SU}(2)_L$ vectors sit in vector multiplets and the remaining gravitini in gravitino multiplets. We equally well could have chosen the $\mathrm{SU}(2)_L\times\mathrm{U}(1)_R$ gauge fields and the other four gravitini to sit in our $\mathcal{N}=4$ gravity multiplet; all eight gravitini are equivalent in this background, as are $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$. Fields such as the broken vectors, however, cannot be placed in a massless $\mathcal{N}=4$ gravity multiplet and would have to be examined by truncating the $\mathcal{N}=8$ theory directly. We shall not do so here.

To avoid confusion, note that this 5D $\mathcal{N}=4$ SUSY does not contain the same 16 supercharges that are preserved in the background. The background preserves 4D $\mathcal{N}=4$, which is half the supercharges of our $\mathcal{N}=4$ multiplet, as well as half of the remaining 16 generators of $\mathcal{N}=8$. The 5D $\mathcal{N}=4$ multiplet is just a useful tool for deriving the equations of motion.

We have the following identifications between Romans' and our notation:

$$\begin{aligned}
&\phi\leftrightarrow\varphi\,, \qquad \xi\leftrightarrow v^{1/3}\,, \qquad g_1\leftrightarrow g\,, \qquad g_2\leftrightarrow\sqrt{2}g\,,\\
&(\Gamma_{45})_{ab}\leftrightarrow\delta_{ab}\,, \qquad T_{ab}\leftrightarrow-\frac{g}{6}W\delta_{ab}\,, \qquad A_{ab}\leftrightarrow\frac{g}{2\sqrt{2}}\frac{\partial W}{\partial\varphi}\delta_{ab}\,.
\end{aligned} \tag{8.1}$$

8.2 Fermion sector

We may now derive the equations of motion for the eight gravitini and the eight spin-1/2 fields χ they couple to. We combine the fields into four sets of complex spinors as before (7.4), and find equations for each system that are identical to those of the GPPZ flow:

$$i\gamma^{\mu\nu\rho}D_\nu\psi_\rho=\frac{g}{2}W\gamma^{\mu\nu}\psi_\nu+\frac{\varphi'}{\sqrt{2}}\left(1+i\gamma^r\right)\gamma^\mu\chi\,, \tag{8.2}$$

$$i\gamma^\mu D_\mu\chi=M(r)\chi-\frac{\varphi'}{\sqrt{2}}\gamma^\mu\left(1-i\gamma^r\right)\psi_\mu\,, \tag{8.3}$$

where again

$$M(r)\equiv-\frac{3}{2}A'-\frac{\varphi''}{\varphi'}\,. \tag{8.4}$$

This universality is presumably a consequence of the $\mathcal{N}=1$ supersymmetry preserved in the background, despite the fact that the 5D multiplets are different in the two cases. We conjecture that this is generally true; it would be interesting to check whether backgrounds with multiple scalars add new features beyond the obvious generalization.

We have already developed the techniques to solve (8.2), (8.3) in section 7. We find the uncoupled equation

$$\left(-e^{-2A}p^2 - \partial_r^2 - 5A'\partial_r - 6A'^2 - 2A'' + M^2 + MA' + M' + \frac{4}{3}\varphi'^2\right)\chi_+ = 0\,, \quad (8.5)$$

which has the solution

$$\chi_+ = v^{a-1/6}(1-v)^{7/4}F(1+a,2+a;2+2a;v)\,\chi_+^{(0)}(p)\,, \quad (8.6)$$

where a was defined below (5.18). One can see that this solution will have the requisite continuous spectrum and mass gap. The corresponding solution for χ_- is

$$\begin{aligned}\chi_- = \frac{1}{3}v^{a-1/6}(1-v)^{5/4}\Big[&(6a(1-v)+2(2-5v))F(1+a,2+a;2+2a;v)+\\ &+3(a+2)v(1-v)F(2+a,3+a;3+2a;v)\Big]\chi_-^{(0)}(p)\,. \end{aligned}\quad (8.7)$$

The behavior of the solutions near the boundary is

$$\chi_+ \sim (1-v)^{3/4}\,, \qquad \chi_- \sim (1-v)^{5/4}\,\log(1-v)\,. \quad (8.8)$$

In this case χ_+ dominates on the boundary, and it scales properly for a field dual to a $\Delta = 5/2$ operator, consistent with its mass $M(r)\to 1/2L$.

8.3 Vector fields

The quadratic action for the $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\times\mathrm{U}(1)_R$ gauge fields in the Coulomb background is

$$e^{-1}\mathcal{L} = -\frac{1}{4}v^{2/3}(F^I_{\mu\nu}F^{\mu\nu I}) - \frac{1}{4}v^{-4/3}f_{\mu\nu}f^{\mu\nu}\,, \quad (8.9)$$

where $v\equiv e^{\sqrt{6}\varphi}$, the $F^I_{\mu\nu}, I=1,\ldots,6$ and $f_{\mu\nu}$ are field strengths for $\mathrm{SO}(4)\cong\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ and $\mathrm{SO}(2)\cong\mathrm{U}(1)_R$, respectively. We have additionally confirmed that the lagrangian (8.9) can be obtained from the $\mathcal{N}=8$ theory directly. We find an equation for the transverse components B_k,

$$\left(e^{-2A}\Box - \partial_r^2 - 2A'\partial_r - b\varphi'\partial_r\right)B_k = 0\,, \quad (8.10)$$

where $b = 4/\sqrt{6}$ for the SO(4) fields and $b=-8/\sqrt{6}$ for the SO(2). We have used the preserved gauge symmetries to choose $B_5 = 0$, and the field equations then require $\partial^i B_i = 0$.

The equation (8.10) resembles that for the graviphoton of the GPPZ flow, but with the term $-b\varphi'\partial_r$ substituting for the mass $m^2 = -2A''$. Let us remove this former term through a field redefinition $B_k \equiv \exp(-b\varphi/2)\hat{B}_k$. We find

$$\left(e^{-2A}\Box - \partial_r^2 - 2A'\partial_r + m_B^2\right)\hat{B}_k = 0\,, \tag{8.11}$$

where

$$\begin{aligned} m_B^2 &\equiv \frac{n\varphi''}{2} + bA'\varphi' + \frac{b^2\varphi'^2}{4}\,, \\ &= \frac{b}{2L^2}\frac{\partial W}{\partial\varphi}\left(\frac{\partial^2 W}{\partial\varphi^2} - \frac{4}{3}W\right) + \frac{b^2}{4L^2}\left(\frac{\partial W}{\partial\varphi}\right)^2 . \end{aligned} \tag{8.12}$$

For a generic choice of W, this need not simplify further. However, one may verify from equation (2.17) that in our case

$$\frac{\partial^2 W}{\partial\varphi^2} - \frac{4}{3}W = \frac{\sqrt{6}}{3}\frac{\partial W}{\partial\varphi}\,. \tag{8.13}$$

One may then show that precisely for $b = 4/\sqrt{6}, -8/\sqrt{6}$ and only for these values,

$$m_B^2 = -2A''\,. \tag{8.14}$$

The equation for these rescaled vectors is thus identical to that for the transverse components of the broken graviphoton in the GPPZ flow, and the same arguments from section 4 imply that their solution is determined by the TT graviton solution, and can again be written in terms of an auxiliary massless scalar f.

8.4 Discussion

It would be interesting to understand the generality of the results we have found coupling the gravity multiplet to the multiplet of the active scalar. The mixing of h^μ_μ and $\tilde{\varphi}$ [43] will be the same for any action of the form (2.1). It is natural to suspect that the form of the gravitino/spin-1/2 system, which was identical in both backgrounds we considered, is a universal consequence of preserving at least four supercharges.

On the other hand, the graviphoton B_μ displays two distinct behaviors in the two examples considered, being Higgsed in the GPPZ case while remaining massless on the Coulomb branch, although the equations of motion turn out to be ultimately equivalent. We are thus led to surmise that for a general supersymmetry RG background, an active hypermultiplet corresponds to a broken $\mathrm{U}(1)_R$, while an active vector (or tensor) multiplet appears when $\mathrm{U}(1)_R$ is preserved. As with the breaking of conformal invariance, the coupling of the bulk fields is not sensitive to whether the breaking is spontaneous or explicit.

Additional support for this hypothesis comes from another supersymmetric one-scalar flow that has appeared in the literature, that of σ alone [16]. As we have seen this field is in a hyper, but unlike the m case, the supersymmetric flow leads to a vev background, which preserves an $SU(3) \subset SU(4)$ but breaks $U(1)_R$. The fact that m and σ lead to different kinds of SUSY backgrounds, the former an operator deformation and the latter a shifted vacuum, is virtually invisible in the action and equations of motion, which treats the fields symmetrically save for factors of 3; see section 3. The two flows share broken $U(1)_R$, and this breaking is accomplished in identical fashion in the two cases through the coupling to the active hypermultiplet. Note that it has been suggested the σ-only flow is unphysical [49], but this presumably does not affect the interplay between the multiplet structure and the symmetries.

One is then led to suspect that the graviphoton always develops a mass $m^2 = -2A''$ when broken, while in the unbroken case it is instead the kinetic terms which are modified. Evidence in favor of such a conjecture is that for general $\mathcal{N} = 2$ SUGRA coupled to matter, scalars from hypermultiplets cannot modify the vector kinetic terms, but scalars from vector and tensor multiplets can [55]. We showed that supersymmetry requires a graviphoton with canonical kinetic terms to have $m^2 = -2A''$ in section 4, so this obtains for all hypermultiplets.

Whether modified kinetic terms are generally equivalent to the mass term, as they were in our case, is an important open question. Interestingly, identities similar to (8.13), which was necessary to derive the equivalence, also hold for all the other Coulomb branch superpotentials from [10]. The factor of $\sqrt{6}/4$ is replaced by a different coefficient in each case. However, the GPPZ flow does not possess such an identity. It is possible that relations like (8.13) are somehow characteristic of Coulomb branch flows only. It is difficult to see how to generalize (8.13) to the case of multiple scalars, however, because the σ-model indices do not seem to match up.

Our results for the anomaly multiplet in the case of preserved R-symmetry imply that the boundary values of a 5D vector multiplet (as well as presumably a tensor multiplet) can couple to a 4D linear multiplet $\mathcal{L}_\alpha$, satisfying $\mathcal{D}_{\dot\alpha}\mathcal{L}_\alpha = 0$ and $\mathcal{D}^\alpha\mathcal{L}_\alpha = \bar{\mathcal{D}}^{\dot\alpha}\bar{\mathcal{L}}_{\dot\alpha}$, in the same way that the gravity and hyper multiplets couple to current and chiral multiplets, respectively. This is not inconsistent with the expectation that a 5D vector can couple to a 4D vector multiplet $\mathcal{V}$, since linear and vector multiplets contain the same degrees of freedom: both have a real scalar and a fermion, while the antisymmetric tensor in the linear multiplet can be identified with the vector field possessing the usual gauge invariance. For a given $\mathcal{V}$, one may always obtain a linear superfield,

$$\mathcal{L}_\alpha = \bar{\mathcal{D}}^2\mathcal{D}_\alpha\mathcal{V}\,, \tag{8.15}$$

a construction familiar from the definition of the linear field strength tensor $\mathcal{W}_\alpha$ in terms of a vector superfield $\mathcal{V}$.

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References

[1] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200];
O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rep.* **323** (2000) 183 [hep-th/9905111].

[2] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *Novel local CFT and exact results on perturbations of $N = 4$ super Yang-Mills from AdS dynamics*, *J. High Energy Phys.* **12** (1998) 022 [hep-th/9810126].

[3] J. Distler and F. Zamora, *Non-supersymmetric conformal field theories from stable anti-de Sitter spaces*, *Adv. Theor. Math. Phys.* **2** (1999) 1405 [hep-th/9810206].

[4] I.R. Klebanov and A.A. Tseytlin, *Asymptotic freedom and infrared behavior in the type-0 string approach to gauge theory*, *Nucl. Phys.* **B 547** (1999) 143 [hep-th/9812089].

[5] J.A. Minahan, *Asymptotic freedom and confinement from type-0 string theory*, *J. High Energy Phys.* **04** (1999) 007 [hep-th/9902074].

[6] A. Kehagias and K. Sfetsos, *On running couplings in gauge theories from type-IIB supergravity*, *Phys. Lett.* **B 454** (1999) 270 [hep-th/9902125]; *On asymptotic freedom and confinement from type-IIB supergravity*, *Phys. Lett.* **B 456** (1999) 22 [hep-th/9903109].

[7] S.S. Gubser, *Dilaton-driven confinement*, hep-th/9902155.

[8] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *Confinement and condensates without fine tuning in supergravity duals of gauge theories*, *J. High Energy Phys.* **05** (1999) 026 [hep-th/9903026].

[9] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Renormalization group flows from holography supersymmetry and a c-theorem*, hep-th/9904017.

[10] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Continuous distributions of D3-branes and gauged supergravity*, *J. High Energy Phys.* **07** (2000) 038 [hep-th/9906194].

[11] A. Brandhuber and K. Sfetsos, *Wilson loops from multicentre and rotating branes, mass gaps and phase structure in gauge theories*, hep-th/9906201.

[12] A. Brandhuber and K. Sfetsos, *Non-standard compactifications with mass gaps and Newton's law*, *J. High Energy Phys.* **10** (1999) 013 [hep-th/9908116].

[13] I. Chepelev and R. Roiban, *A note on correlation functions in AdS_5/SYM_4 correspondence on the Coulomb branch*, *Phys. Lett.* **B 462** (1999) 74 [hep-th/9906224].

[14] K. Behrndt, *Domain walls of $D = 5$ supergravity and fixpoints of $N = 1$ super Yang-Mills*, *Nucl. Phys.* **B 573** (2000) 127 [hep-th/9907070].

[15] S.B. Giddings and S.F. Ross, *D3-brane shells to black branes on the Coulomb branch*, *Phys. Rev.* **D 61** (2000) 024036 [hep-th/9907204].

[16] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *The supergravity dual of $N = 1$ super Yang-Mills theory*, *Nucl. Phys.* **B 569** (2000) 451 [hep-th/9909047].

[17] I. Bakas and K. Sfetsos, *States and curves of five-dimensional gauged supergravity*, *Nucl. Phys.* **B 573** (2000) 768 [hep-th/9909041];
I. Bakas, A. Brandhuber and K. Sfetsos, *Domain walls of gauged supergravity, m-branes and algebraic curves*, hep-th/9912132.

[18] M. Cvetič, S.S. Gubser, H. Lu and C.N. Pope, *Symmetric potentials of gauged supergravities in diverse dimensions and Coulomb branch of gauge theories*, *Phys. Rev.* **D 62** (2000) 086003 [hep-th/9909121].

[19] P. Kraus, *Dynamics of anti-de Sitter domain walls*, *J. High Energy Phys.* **12** (1999) 011 [hep-th/9910149].

[20] J. Distler and F. Zamora, *Chiral symmetry breaking in the AdS/CFT correspondence*, *J. High Energy Phys.* **05** (2000) 005 [hep-th/9911040].

[21] C.V. Johnson, A.W. Peet and J. Polchinski, *Gauge theory and the excision of repulson singularities*, *Phys. Rev.* **D 61** (2000) 086001 [hep-th/9911161].

[22] K. Behrndt and M. Cvetič, *Anti-de Sitter vacua of gauged supergravities with 8 supercharges*, *Phys. Rev.* **D 61** (2000) 101901 [hep-th/0001159].

[23] J. Polchinski and M.J. Strassler, *The string dual of a confining four-dimensional gauge theory*, hep-th/0003136.

[24] K. Pilch and N.P. Warner, *$N = 2$ supersymmetric RG flows and the IIB dilaton*, *Nucl. Phys.* **B 594** (2001) 209 [hep-th/0004063].

[25] A. Brandhuber and K. Sfetsos, *An $N = 2$ gauge theory and its supergravity dual*, *Phys. Lett.* **B 488** (2000) 373 [hep-th/0004148].

[26] N. Evans and M. Petrini, *AdS RG-flow and the super Yang-Mills cascade*, *Nucl. Phys.* **B 592** (2000) 129 [hep-th/0006048].

[27] K. Pilch and N.P. Warner, *$N = 1$ supersymmetric renormalization group flows from IIB supergravity*, hep-th/0006066.

[28] I.R. Klebanov and M.J. Strassler, *Supergravity and a confining gauge theory: duality cascades and χSB-resolution of naked singularities*, *J. High Energy Phys.* **08** (2000) 052 [hep-th/0007191].

[29] J.M. Maldacena and C. Nunez, *Towards the large-N limit of pure $N = 1$ super Yang-Mills*, *Phys. Rev. Lett.* **86** (2001) 588 [hep-th/0008001].

[30] A. Buchel, A.W. Peet and J. Polchinski, *Gauge dual and noncommutative extension of an $N = 2$ supergravity solution*, *Phys. Rev.* **D 63** (2001) 044009 [hep-th/0008076].

[31] N. Evans, C.V. Johnson and M. Petrini, *The enhancon and $N = 2$ gauge theory/gravity RG flows*, *J. High Energy Phys.* **10** (2000) 022 [hep-th/0008081].

[32] K. Behrndt, C. Herrmann, J. Louis and S. Thomas, *Domain walls in five dimensional supergravity with non-trivial hypermultiplets*, *J. High Energy Phys.* **01** (2001) 011 [hep-th/0008112].

[33] E. Alvarez and C. Gomez, *Geometric holography, the renormalization group and the c-theorem*, *Nucl. Phys.* **B 541** (1999) 441 [hep-th/9807226].

[34] V. Balasubramanian and P. Kraus, *Spacetime and the holographic renormalization group*, *Phys. Rev. Lett.* **83** (1999) 3605 [hep-th/9903190].

[35] K. Skenderis and P.K. Townsend, *Gravitational stability and renormalization-group flow*, *Phys. Lett.* **B 468** (1999) 46 [hep-th/9909070].

[36] J. de Boer, E. Verlinde and H. Verlinde, *On the holographic renormalization group*, *J. High Energy Phys.* **08** (2000) 003 [hep-th/9912012].

[37] M. Li, *A note on relation between holographic RG equation and Polchinski's RG equation*, *Nucl. Phys.* **B 579** (2000) 525 [hep-th/0001193].

[38] D. Anselmi, L. Girardello, M. Porrati and A. Zaffaroni, *A note on the holographic beta and c functions*, *Phys. Lett.* **B 481** (2000) 346 [hep-th/0002066].

[39] V. Sahakian, *Comments on D-branes and the renormalization group*, *J. High Energy Phys.* **05** (2000) 011 [hep-th/0002126].

[40] V. Balasubramanian, E. Gimon and D. Minic, *Consistency conditions for holographic duality*, *J. High Energy Phys.* **05** (2000) 014 [hep-th/0003147].

[41] M. Pernici, K. Pilch and P. van Nieuwenhuizen, *Gauged $N = 8$ $D = 5$ supergravity*, *Nucl. Phys.* **B 259** (1985) 460.

[42] M. Günaydin, L.J. Romans and N.P. Warner, *Gauged $N = 8$ supergravity in five-dimensions*, *Phys. Lett.* **B 154** (1985) 268.

[43] O. De Wolfe and D.Z. Freedman, *Notes on fluctuations and correlation functions in holographic renormalization group flows*, hep-th/0002226.

[44] G. Arutyunov, S. Frolov and S. Theisen, *A note on gravity-scalar fluctuations in holographic RG flow geometries*, *Phys. Lett.* **B 484** (2000) 295 [hep-th/0003116].

[45] L. Andrianopoli, R. D'Auria, S. Ferrara, P. Fré and M. Trigiante, *R-R scalars, U-duality and solvable Lie algebras*, *Nucl. Phys.* **B 496** (1997) 617 [hep-th/9611014].

[46] M. Petrini and A. Zaffaroni, *The holographic RG flow to conformal and non-conformal theory*, hep-th/0002172.

[47] P. West, *Introduction to supersymmetry and supergravity*, World Scientific, Singapore 1990;
M.F. Sohnius and P.C. West, *An alternative minimal off-shell version of $N = 1$ supergravity*, *Phys. Lett.* **B 105** (1981) 353.

[48] V. Balasubramanian, E. Gimon, D. Minic and J. Rahmfeld, *Four dimensional conformal supergravity from AdS space*, hep-th/0007211.

[49] S.S. Gubser, *Curvature singularities: the good, the bad and the naked*, hep-th/0002160.

[50] E. Shuster, *Killing spinors and supersymmetry on AdS*, *Nucl. Phys.* **B 554** (1999) 198 [hep-th/9902129].

[51] M. Günaydin, L.J. Romans and N.P. Warner, *Compact and noncompact gauged supergravity theories in five-dimensions*, *Nucl. Phys.* **B 272** (1986) 598.

[52] H. Lu, C.N. Pope and J. Rahmfeld, *A construction of killing spinors on S^n*, *J. Math. Phys.* **40** (1999) 4518 [hep-th/9805151].

[53] S. Ferrara and S. Sabharwal, *Quaternionic manifolds for type-II superstring vacua of Calabi-Yau spaces*, *Nucl. Phys.* **B 332** (1990) 317.

[54] M. Bodner and A.C. Cadavid, *Dimensional reduction of type-IIB supergravity and exceptional quaternionic manifolds*, *Class. and Quant. Grav.* **7** (1990) 829.

[55] A. Ceresole and G. Dall'Agata, *General matter coupled $N = 2$, $D = 5$ gauged supergravity*, *Nucl. Phys.* **B 585** (2000) 143 [hep-th/0004111].

[56] E. Cremmer, *Supergravities in 5 dimensions*, in *Superspace and supergravity*, S.W. Hawking and M. Roček eds., Cambridge University Press, Cambridge, 1981.

[57] E.T. Akhmedov, *D-instantons probing D3-branes and the AdS/CFT correspondence*, *Phys. Rev.* **D 59** (1999) 101901 [hep-th/9812038].

[58] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York 1962.

[59] O. DeWolfe, D.Z. Freedman, S.S. Gubser and A. Karch, *Modeling the fifth dimension with scalars and gravity*, *Phys. Rev.* **D 62** (2000) 046008 [hep-th/9909134].

[60] L. Randall and R. Sundrum, *An alternative to compactification*, *Phys. Rev. Lett.* **83** (1999) 4690 [hep-th/9906064]; *A large mass hierarchy from a small extra dimension*, *Phys. Rev. Lett.* **83** (1999) 3370 [hep-ph/9905221].

[61] A. Erdelyi, *Higher transcendental functions*, vol. 1, McGraw-Hill, 1953.

[62] C. Imbimbo, A. Schwimmer, S. Theisen and S. Yankielowicz, *Diffeomorphisms and holographic anomalies*, *Class. and Quant. Grav.* **17** (2000) 1129 [hep-th/9910267].

[63] R. Penrose and W. Rindler, *Spinors and spacetime*, vol. 2, Cambridge University Press, Cambridge 1986, ch. 9.

[64] J.D. Brown and M. Henneaux, *Central charges in the canonical realization of asymptotic symmetries: an example from three-dimensional gravity*, *Comm. Math. Phys.* **104** (1986) 207.

[65] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109];
E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[66] D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Correlation functions in the CFT_d/AdS_{d+1} correspondence*, *Nucl. Phys.* **B 546** (1999) 96 [hep-th/9804058].

[67] P. Minces and V.O. Rivelles, *Scalar field theory in the AdS/CFT correspondence revisited*, *Nucl. Phys.* **B 572** (2000) 651 [hep-th/9907079].

[68] L.J. Romans, *Gauged $N = 4$ supergravities in five-dimensions and their magnetovac backgrounds*, *Nucl. Phys.* **B 267** (1986) 433.